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ABSTRACT

Problem Solving in Mathematics (PSM), a program of problem-solving lessons for grades 4-9, is designed to be integrated into the regular mathematics program. Although the content objectives of the lessons are similar to those of most textbooks, PSM emphasizes problem-solving skills and a variety of instructional strategies which foster the development of such skills. These strategies include direct instruction, guided discovery, laboratory work, small-group discussions, nondirective instruction, and individual work. This PSM book for grade 8 consists of 11 parts. The first 10 parts contain activities focusing on: five basic problem-solving skills; drill and practice for whole numbers, fractions, and decimals; variation; integer sense; equation solving; measuring with protractors; geometry; calculators; percent estimation; and probability. The last section contains 16 challenge problems in which students select the problem-solving method to use. Each of these parts has an overview teacher commentary, which includes information on PSM philosophy and suggestions as to how the activities within the part should be used. In addition, each activity includes mathematics teaching objectives, problem-solving skills students might use, materials needed, comments and suggestions, and answers. General information on the PSM program and PSM strategies are provided in an introduction. (JN)

Lane County Mathematics Project

Grade 8

PROBLEM SOLVING IN MATHEMATICS

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With Teacher Commentary and Answers

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CONTENTS

Introduction..... i-xi

I. Getting Started 3-34

Guess And Check 5
Look For A Pattern 11
Make A Systematic List 17
Make And Use A Drawing Or Model 23
Eliminate Possibilities 29

II. Drill And Practice 37-77

Complete The Problem 39
More Than One Solution 41
Decimal Digit Draw Activities 43
Distributive Property 47
Create A Problem 53
Patterns 55
Find A Path 57
The Best Guess (Money) 59
The Best Guess (Decimals) .. 65
The Best Guess (Fractions) . 71

III. Variation 81-105

Hamburgers For Slim 83
Formulas 91
Believe It Or Not Formulas . 97
Cars And Bikes 101

IV. Integer Sense 109-126

Thermometer Readings 111
Negative Numbers 117
What's The Sum? 121
A Tricky Pattern 125

V. Equation Solving 129-144

Puzzle Problems 131
The Hidden Solution 135
Arthur's Dilemma 137
Another Dilemma 139
Unusual Equations 143

VI. Protractor Experiments . 147-163

Protractor Practice 149
More Protractor Practice ... 153
What's The Sum? 157
Water-Wheel Geometry 161

VII. Investigations In Geometry 167-192

Patterns In Geometry 169
Quadrilaterals 173
Snipping Quadrilaterals 177
Oil Wells 179
Congruent Triangles 181
Drawing Triangles 185
Pythagoras 189

VIII. Calculator..... 195-208

A Number Times Itself 197
Best Estimate 199
Discovering Patterns 201
Missing Parts 203
Calculator Puzzlers 205
I'm Thinking Of A Number 207

IX. Percent Estimation 211-239

Meaning Of Percent 213
One-Percent Method 219
Percents And Fractions 223
Estimation 231
Short Stories 237

X. Probability 243-252

Dice 245
Roll That Cube 247
Coin Toss 249
The Gum Machine 251

XI. Challenges 255-292

Magic Circles 259
Use All Nine 261
Pool 263
Logical Thinking 265
A Game Of Cards 267
Pizza Puzzles 269
Magic Triangle 271
Dominoes 275
Game Of Darts 277
Happy Numbers 279
The Last Digit 281
Birthday Party 283
Where Is The Monkey? 285
A Row Of Pennies 287
Mr. Wizard's Marble Trick 289
Chips Ahoy 291

INTRODUCTION

What is PSM?

PROBLEM SOLVING IN MATHEMATICS is a program of problem-solving lessons and teaching techniques for grades 4–8 and (9) algebra. Each grade-level book contains approximately 80 lessons and a teacher's commentary with teaching suggestions and answer key for each lesson. *Problem Solving in Mathematics* is not intended to be a complete mathematics program by itself. Neither is it supplementary in the sense of being extra credit or to be done on special days. Rather, it is designed to be integrated into the regular mathematics program. Many of the problem-solving activities fit into the usual topics of whole numbers, fractions, decimals, percents, or equation solving. Each book begins with lessons that teach several problem-solving skills. Drill and practice, grade-level topics, and challenge activities using these problem-solving skills complete the book.

PROBLEM SOLVING IN MATHEMATICS is designed for use with all pupils in grades 4–8 and (9) algebra. At-grade-level pupils will be able to do the activities as they are. More advanced pupils may solve the problems and then extend their learning by using new data or creating new problems of a similar nature. Low achievers, often identified as such only because they haven't reached certain computational levels, should be able to do the work in PSM with minor modifications. The teacher may wish to work with these pupils at a slower pace using more explanations and presenting the material in smaller doses.

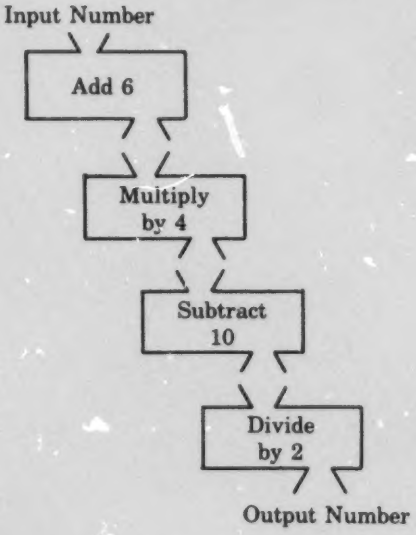
[Additional problems appropriate for low achievers are contained in the *Alternative Problem Solving in Mathematics* book. Many of the activities in that book are similar to those in the regular books except that the math computation and length of time needed for completion are scaled down. The activities are generally appropriate for pupils in grades 4–6.]

Why Teach Problem Solving?

Problem solving is an ability people need throughout life. Pupils have many problems with varying degrees of complexity. Problems arise as they attempt to understand concepts, see relationships, acquire skills, and get along with their peers, parents, and teachers. Adults have problems, many of which are associated with making a living, coping with the energy crisis, living in a nation with peoples from different cultural backgrounds, and preserving the environment. Since problems are so central to living, educators need to be concerned about the growth their pupils make in tackling problems.

What Is a Problem?

MACHINE HOOK-UPS



It is highly recommended that teachers intending to use *Problem Solving in Mathematics* receive training in implementing the program. The *In-Service Guide* contains much of this valuable material. In addition, in-service audio cassette tapes are available. These provide indepth guidance on using the PSM grade-level books and an overall explanation of how to implement the whole program. The tapes are available for loan upon request. Please contact Dale Seymour Publications, Box 10888, Palo Alto, CA 94303 for further information about the tapes and other possible in-service opportunities.

	Input Number	Output Number
a.	4	
b.	8	
c.	12	
d.		39
e.		47
f.		61

Suppose a 6th grader were asked to fill in the missing output blanks for *a*, *b*, and *c* in the table. Would this be a problem for him? Probably not, since all he would need to do is to follow the directions. Suppose a 2nd-year algebra student were asked to fill in the missing input blank for *d*. Would this be a problem? Probably not, since she would write the suggested equation,

$$\frac{4(x + 6) - 10}{2} = 39$$

and then solve it for the input. Now suppose the 6th grader were asked to fill in the input for *d*, would this be a problem for him? Probably it *would* be. He has no directions for getting the answer. However, if he has the desire, it is within his power to find the answer. What might he do? Here are some possibilities:

1. He might make *guesses*, do *checking*, and then make refinements until he gets the answer.
2. He might fill in the output numbers that correspond to the input numbers for *a*, *b*, and *c*.

	Input Number	Output Number
a.	4	
b.	8	
c.	12	
d.		39

and then observe this pattern:
For an increase of 4 for the input, the output is increased by 8.
Such an observation should lead quickly to the required input of 16.

3. He might start with the output and *work backwards* through the machine hook-up using the inverse (or opposite) operations.

For this pupil, there was no "ready-made" way for him to find the answer, but most motivated 6th-grade pupils would find a way.

A *problem*, then, is a situation in which an individual or group accepts the challenge of performing a task for which there is no immediately obvious way to determine a solution. Frequently, the problem can be approached in many ways. Occasionally, the resulting investigations are nonproductive. Sometimes they are so productive as to lead to many different solutions or suggest more problems than they solve.

What Does Problem Solving Involve?

Problem solving requires the use of many *skills*. Usually these skills need to be used in certain combinations before a problem is solved. A combination of skills used in working toward the solution of a problem can be referred to as a *strategy*. A successful strategy requires the individual or group to generate the information needed for solving the problem. A considerable amount of creativity can be involved in generating this information.

What Problem-Solving Skills Are Used in PSM?

Skills are the building blocks used in solving a problem. The pupil materials in the PSM book afford many opportunities to emphasize problem-solving skills. A listing of these skills is given below.

THE PSM CLASSIFIED LIST
OF PROBLEM-SOLVING SKILLS

A. Problem Discovery, Formulation
   1. State the problem in your own words.
   2. Clarify the problem through careful reading and by asking questions.
   3. Visualize an object from its drawing or description.
   4. Follow written and/or oral directions.

B. Seeking Information
   5. Collect data needed to solve the problem.
   6. Share data and results with other persons.
   7. Listen to persons who have relevant knowledge and experiences to share.
   8. Search printed matter for needed information.
   9. Make necessary measurements for obtaining a solution.
   10. Record solution possibilities or attempts.
   11. Recall and list related information and knowledge.

C. Analyzing Information
   12. Eliminate extraneous information.
   13. Find likenesses and differences and make comparisons.
   14. Classify objects or concepts.
   15. Make and use a drawing or model.
   16. Make and/or use a systematic list or table.
   17. Make and/or use a graph.
   18. Look for patterns and/or properties.
   19. Use mathematical symbols to describe situations.
   20. Break a problem into manageable parts.

D. Solve—Putting It Together—Synthesis
   21. Make predictions, conjectures, and/or generalizations based upon data.
   22. Make decisions based upon data.
   23. Make necessary computations needed for the solution.
   24. Determine limits and/or eliminate possibilities.
   25. Make reasonable estimates.
   26. Guess, check, and refine.
   27. Solve an easier but related problem. Study solution process for clues.
   28. Change a problem into one you can solve. (Simplify the problem.)
   29. Satisfy one condition at a time.
   30. Look at problem situation from different points of view.
   31. Reason from what you already know. (Deduce.)
   32. Work backwards.
   33. Check calculated answers by making approximations.
   34. Detect and correct errors.
   35. Make necessary measurements for checking a solution.
   36. Identify problem situation in which a solution is not possible.
   37. Revise the conditions of a problem so a solution is possible.

E. Looking Back—Consolidating Gains
   38. Explain how you solved a problem.
   39. Make explanations based upon data.
   40. Solve a problem using a different method.
   41. Find another answer when more than one is possible.
   42. Double check solutions by using some formal reasoning method (mathematical proof).
   43. Study the solution process.
   44. Find or invent other problems which can be solved by certain solution procedures.
   45. Generalize a problem solution so as to include other solutions.

F. Looking Ahead—Formulating New Problems
   46. Create new problems by varying a given one.

What Are Some Examples of Problem-Solving Strategies?

Since strategies are a combination of skills, a listing (if it were possible) would be even more cumbersome than the list of skills. Examples of some strategies that might be used in the "Machine Hook-Ups" problem follow:

Strategy 1. *Guess* the input; *check* by computing the output number for your guess; if guess does not give the desired output, note the direction of error; *refine* the guess; compute; continue making refinements until the correct output results.

Strategy 2. *Observe* the *patterns* suggested by the input and output numbers for the *a*, *b*, *c* entries in the table; *predict* additional output and input numbers by extending both patterns; *check* the predicted input for the *d* entry by computing.

Strategy 3. *Study* the operations suggested in the machine hook-up; *work backwards* through the machine *using previous knowledge* about inverse operations.

An awareness of the strategies being used to solve a problem is probably the most important step in the development of a pupil's problem-solving abilities.

What is the Instructional Approach Used in PSM?

The content objectives of the lessons are similar to those of most textbooks. The difference is in the approach used. First, a wider variety of problem-solving skills is emphasized in the materials than in most texts. Second, different styles of teaching such as direct instruction, guided discovery, laboratory work, small-group discussions, nondirective instruction, and individual work all have a role to play in problem-solving instruction.

Most texts employ direct instruction almost exclusively, whereas similar lessons in PSM are patterned after a guided discovery approach. Also, an attempt is made in the materials to use intuitive approaches extensively before teaching formal algorithms. Each of the following is an integral part of the instructional approach to problem solving.

A. TEACH PROBLEM-SOLVING SKILLS DIRECTLY

Problem-solving skills such as "follow directions," "listen," and "correct errors" are skills teachers expect pupils to master. Yet, such skills as "guess and check," "make a systematic list," "look for a pattern," or "change a problem into one you can solve" are seldom made the object of direct instruction. These skills, as well as many more, need emphasis. Detailed examples for teaching these skills early in the school year are given in the commentaries to the *Getting Started* activities.

B. INCORPORATE A PROBLEM-SOLVING APPROACH WHEN TEACHING TOPICS IN THE COURSE OF STUDY

Drill and practice activities. Each PSM book includes many pages of drill and practice at the problem-solving level. These pages, along with the *Getting Started* section, are easy for pupils and teachers to get into and should be started early in the school year.

Laboratory activities and investigations involving mathematical applications and readiness activities. Readiness activities from such mathematical strands as geometry, number theory, and probability are included in each book. For example, area explorations are used in grade 4 as the initial stage in the teaching of the multiplication and division algorithms and fraction concepts.

Teaching mathematical concepts, generalizations, and processes. Each book includes two or more sections on grade-level content topics. For the most part, these topics are developmental in nature and usually need to be supplemented with practice pages selected from a textbook.

C. PROVIDE MANY OPPORTUNITIES FOR PUPILS TO USE THEIR OWN PROBLEM-SOLVING STRATEGIES

One section of each book includes a collection of challenge activities which provide opportunities for emphasizing problem-solving strategies. Generally, instruction should be nondirective, but at times suggestions may need to be given. If possible, these suggestions should be made in the form of alternatives to be explored rather than hints to be followed.

D. CREATE A CLASSROOM ATMOSPHERE IN WHICH OPENNESS AND CREATIVITY CAN OCCUR

Such a classroom climate should develop if the considerations mentioned in A, B, and C are followed. Some specific suggestions to keep in mind as the materials are used are:

- Set an example by solving problems and by sharing these experiences with the pupils.
- Reduce anxiety by encouraging communication and cooperation. On frequent occasions problems might be investigated using a cooperative mode of instruction along with brainstorming sessions.
- Encourage pupils in their efforts to solve a problem by indicating that their strategies are worth trying and by providing them with sufficient time to investigate the problem; stress the value of the procedures pupils use.
- Use pupils' ideas (including their mistakes) in solving problems and developing lessons.
- Ask probing questions which make use of words and phrases such as
  I wonder if . . . .
  Do you suppose that . . . .
  What happens if . . . .
  How could we find out . . . .
  Is it possible that . . . .
- Reinforce the asking of probing questions by pupils as they search for increased understanding. Pupils seldom are skilled at seeking probing questions but they can be taught to do so. If instruction is successful, questions of the type, "What should I do now?," will be addressed to themselves rather than to the teacher.

What Are the Parts of Each PSM Book?

PROBLEM SOLVING IN MATHEMATICS

Grade 4	Grade 5	Grade 6	Grade 7	Grade 8	Grade 9
Getting Started	Getting Started	Getting Started	Getting Started	Getting Started	Getting Started
Place Value Drill and Practice	Whole Number Drill and Practice	Drill and Practice	Drill and Practice-Whole Numbers	Drill and Practice	Algebraic Concepts and Patterns
Whole Number Drill and Practice	Story Problems	Story Problems	Drill and Practice-Fractions	Variation	Algebraic Explanations
Multiplication and Division Concepts	Fractions	Fractions	Drill and Practice-Decimals	Integer Sense	Equation Solving
Fraction Concepts	Geometry	Geometry	Percent Sense	Equation Solving	Word Problems
Two-digit Multiplication	Decimals	Decimals	Factors, Multiples, and Primes	Protractor Experiments	Binomials
Geometry	Probability	Probability	Measurement-Volume, Area, Perimeter	Investigations in Geometry	Graphs and Equations
Rectangles and Division	Estimation with Calculators	Challenges	Probability	Calculator	Graph Investigations
Challenges	Challenges		Challenges	Percent Estimation	Systems of Linear Equations
				Probability	Challenges
				Challenges	

Notice that the above chart is only a scope of PSM—not a scope and sequence. In general, no sequence of topics is suggested with the exceptions that *Getting Started* activities must come early in the school year and *Challenge* activities are usually deferred until later in the year.

Getting Started Several problem-solving skills are presented in the *Getting Started* section of each grade level. Hopefully, by concentrating on these skills during the first few weeks of school pupils will have confidence in applying them to problems that occur later on. In presenting these skills, a direct mode of instruction is recommended. Since the emphasis needs to be on the problem-solving skill used to find the solution, about ten to twelve minutes per day are needed to present a problem.

Drill And Practice No sequence is implied by the order of activities included in these sections. They can be used throughout the year but are especially appropriate near the beginning of the year when the initial chapters in the textbook emphasize review. Most of the activities are not intended to develop any particular concept. Rather, they are drill and practice lessons with a problem-solving flavor.

Challenges Fifteen or more challenge problems are included in each book. In general, these should be used only after *Getting Started* activities have been completed and pupils have had some successful problem-solving experiences.

Many of the other sections in PSM are intended to focus on particular grade-level content. The purpose is to provide intuitive background for certain topics. A more extensive textbook treatment usually will need to follow the intuitive development.

Teacher Commentaries Each section of a PSM book has an overview teacher commentary. The overview commentary usually includes some philosophy and some suggestions as to how the activities within the section should be used. Also, every pupil page in PSM has a teacher commentary on the back of the lesson. Included here are mathematics teaching objectives, problem-solving skills pupils might use, materials needed, comments and suggestions, and answers.

How Often Should Instruction Be Focused on Problem Solving?

Some class time should be given to problem solving nearly every day. On some days an entire class period might be spent on problem-solving activities; on others, only 8 to 10 minutes. Not all the activities need to be selected from PSM. Your textbook may contain ideas. Certainly you can create some of your own. Many companies now have published excellent materials which can be used as sources for problem-solving ideas. Frequently, short periods of time should be used for identifying and comparing problem-solving skills and strategies used in solving problems.

How Can I Use These Materials When I Can't Even Finish What's in the Regular Textbook?

This is a common concern. But PSM is not intended to be an "add-on" program. Instead, much of PSM can replace material in the textbook. Correlation charts can be made suggesting how PSM can be integrated into the course of study or with the adopted text. Also, certain textbook companies have correlated their tests with the PSM materials.

Can the Materials Be Duplicated?

The pupil lessons may be copied for students. Each pupil lesson may be used as an overhead projector transparency master or as a blackline duplicator master. Sometimes the teacher may want to project one problem at a time for pupils to focus their attentions on. Other times, the teacher might want to duplicate a lesson for individual or small group work. Permission to duplicate pupil lesson pages for classroom use is given by the publisher.

How Can a Teacher Tell Whether Pupils Are Developing and Extending Their Problem-Solving Abilities?

Presently, reliable paper and pencil tests for measuring problem-solving abilities are not available. Teachers, however, can detect problem-solving growth by observing such pupil behaviors as

- identifying the problem-solving skills being used.
- giving accounts of successful strategies used in working on problems.
- insisting on understanding the topics being studied.
- persisting while solving difficult problems.
- working with others to solve problems.
- bringing in problems for class members and teachers to solve.
- inventing new problems by changing problems previously solved.

What Evidence Is There of the Effectiveness of PSM?

Although no carefully controlled longitudinal study has been made, evaluation studies do indicate that pupils, teachers, and parents like the materials. Scores on standardized mathematics achievement tests show that pupils are registering greater gains than expected on all parts of the test, including computation. Significant gains were made on special problem-solving skills tests which were given at the beginning and end of a school year.

Also, when selected materials were used exclusively over a period of several weeks with 6th-grade classes, significant gains were made on the word-problem portion of the standardized test. In general, the greater gains occurred in those classrooms where the materials were used as specified in the teacher commentaries and in-service materials.

Teachers have indicated that problem-solving skills such as *look for a pattern*, *eliminate possibilities*, and *guess and check* do carry over to other subjects such as Social Studies, Language Arts, and Science. Also, the materials seem to be working with many pupils who have not been especially successful in mathematics. And finally, many teachers report that PSM has caused them to make changes in their teaching style.

Why Is It Best to Have Whole-Staff Commitment?

Improving pupils' abilities to solve problems is not a short-range goal. In general, efforts must be made over a long period of time if permanent changes are to result. Ideally, then, the teaching staff for at least three successive grade levels should commit themselves to using PSM with their pupils. Also, if others are involved, this will allow for opportunities to plan together and to share experiences.

How Much In-Service Is Needed?

A teacher who understands the meaning of problem solving and is comfortable with the different styles of teaching it requires could get by with self in-service by carefully studying the section and page commentaries in a grade level book. The different styles of teaching required include direct instruction, guided discovery, laboratory work, small group instruction, individual work, and non-directive instructions. The teacher would find the audio tapes for each book and the *In-Service Guide* a valuable resource and even a time saver.

If a school staff decides to emphasize problem solving in all grade levels where PSM books are available, in-service sessions should be led by someone who has used the materials in the intended way. For more information on this in-service see the *In-Service Guide*.

What Materials Are Needed?

REQUIRED MATERIALS	Grade 4	5	6	7	8	9
blank cards	X	X	X	X	X	X
bottle caps or markers	X			X		
calendar						X
calculators (optional for some activities)	X	X	X	X	X	X
cm squared paper, strips and singles						X
coins				X	X	
colored construction paper (circle fractions)	X	X				
cubes		X	X	X		X
cubes with red, yellow and green faces					X	
Cuisenaire rods (orange and white) or strips of paper		X				
dice (blank wooden or foam, for special dice)	X					
dice, regular (average 2 per student)	X	X	X	X	X	
geoboards, rubber bands, and record paper	X		X			
graph paper or cm squared paper				X		X
grid paper (1")			X			
metric rulers					X	X
phone books, newspapers, magazines		X		X		
protractors and compasses					X	
scissors	X	X	X	X		
spinners (2 teacher-made)			X			
tangrams	X					
tape measures				X		
thumbtacks (10 per pair of students)		X				
tile	X		X			
tongue depressors	X					
uncooked spaghetti or paper strips			X			

RECOMMENDED MATERIALS	Grade 4	5	6	7	8	9
adding machine tape				X		
centimetre rulers			X	X		
colored pens, pencils or crayons		X				
coins, toy or real	X					
coins (two and one-half)						X
cubes					X	
demonstration ruler for overhead		X				
dominoes					X	
geoboard, transparent (for overhead)	X		X			
money - 20 $1.00 bills per student			X			
moveable markers		X	X		X	
octahedral die for extension activity				X		
overhead projector	X	X	X	X	X	X
place value frame and markers			X			
straws, uncooked spaghetti, or toothpicks		X				
transparent circle fractions for overhead	X	X				

Where Can I Find Other Problem Solving Materials?

RESOURCE BIBLIOGRAPHY

The number in parentheses refers to the list of publishers on the next page.

For students and teachers:

AFTER MATH, BOOKS I-IV by Dale Seymour, et al.
Puzzles to solve -- some of them non-mathematical. (1)

AHA, INSIGHT by Martin Gardner
Puzzles to solve -- many of them non-mathematical. (3)

THE BOOK OF THINK by Marilyn Burns
Situations leading to a problem-solving investigation. (1)

CALCULATOR ACTIVITIES FOR THE CLASSROOM, BOOKS 1 & 2 by George Immerzeel and Earl Ockenga
Calculator activities using problem solving. (1)

GEOMETRY AND VISUALIZATION by Mathematics Resource Project
Resource materials for geometry. (1)

GOOD TIMES MATH EVENT BOOK by Marilyn Burns
Situations leading to a problem-solving investigation. (1)

FAVORITE PROBLEMS by Dale Seymour
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Publisher's List

1. Creative Publications, 3977 E Bayshore Rd, PO Box 10328, Palo Alto, CA 94303
2. Creative Teaching Associates, PO Box [illegible]714, Fresno, CA 93727
3. Dale Seymour Publications, PO Box 10[illegible], Palo Alto, CA 94303
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7. National Council of Teachers of Mathematics, 1906 Association Dr, Reston, VA 22091
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9. Oregon Department of Education, 700 Pringle Parkway SE, Salem, OR 97310

Grade 8

I. GETTING STARTED

I. GETTING STARTED

Teachers usually are successful at teaching skills in mathematics. Besides computation skills, they emphasize skills in following directions, listening, detecting errors, explaining, recording, comparing, measuring, sharing, ... They (You!) can also teach problem-solving skills. This section is designed to help teachers teach and pupils learn specific problem-solving skills.

Some Problem-Solving Skills

Five common but powerful problem-solving skills are introduced in this section. They are:

- guess and check
- look for a pattern
- make a systematic list
- make and use a drawing or model
- eliminate possibilities

Pupils might use other skills to solve the problems. They can be praised for their insight, but it is usually a good idea to limit the list of skills taught during the first few lessons. More problem-solving skills will occur in other sections.

An Important DON'T

When you read the episodes that follow in this Getting Started section, notice how the lessons are teacher directed. The main purpose is to teach the problem-solving skills. Teachers should stress the skills verbally and write them on the board. Don't just ditto these activities and hand them out to be worked. Teacher direction through questions, summaries, praise, etc., is most important for teaching the problem-solving skills in this section. We want pupils to focus on specific skills which will be used often in all the sections. Later, in the Challenges section, pupils will be working more independently.

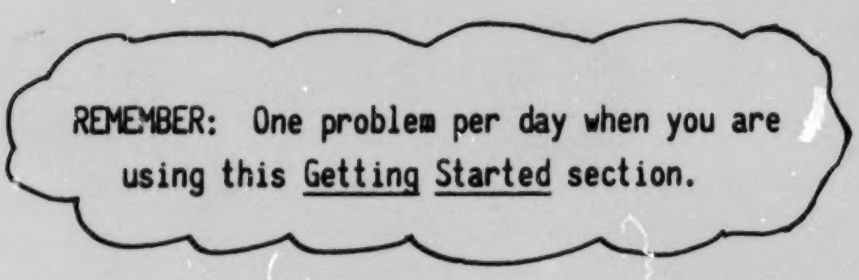
Using The Activities

If you heed the important Don't on the previous page, you are on your way to success! The problems here should fit right in with your required course of study as they use whole number skills, elementary geometry, and money concepts. In most cases, pupils will have the prerequisites for the problems in this section, although you might want to check over each problem to be sure.

No special materials are required although markers, coins, and dice are helpful for some of the problems. The large type used for the problems makes them easier to read if they are shown on an overhead screen. In most cases, pupils can easily copy the problem from the overhead. At other times you might copy the problem onto the chalkboard.

When And How Many

The Getting Started section should be used at the beginning of the year as it builds background in problem-solving skills for the other sections. As the format indicates, only one problem per day should be used. Each should take less than twelve minutes of classtime if the direct mode of instruction is used. The remainder of the period is used for a lesson from the textbook or perhaps an activity from the Drill And Practice section of these materials.



Guess And Check

The episode that follows shows how one teacher teaches the skill of guess and check. Notice that she very closely directs the instruction and constantly uses the terminology.

Ms. J: Who am I? If you multiply me by 15 and then add 27, the result is 147. I wonder if the number is 5. Is it? (Waits for hands.) Tim?

Tim: Nope!

Ms. J: How do you know?

Tim: 'Cause 5 multiplied by 15 is 75, add 27 gives 102.

Ms. J: By checking my guess you found out it was off. Well, is the number 9? Mary?

Mary: (Thinking out loud.) Nine times 15 is 135, add 27 is 162. No--too big.

Ms. J: Nine is too big? What can you say about 5?

Tim: It was too small.

Ms. J: Guessing and checking helped you decide 5 is too small and 9 is too big. Let's refine the guesses. Refine means to make better guesses. Can you make a better guess? (Some puzzled, some thinking, some hands.) Larry?

Larry: The answer probably should be closer to 9 than 5. Try 8.

Sue: Yes, that works. Eight times 15 is 120, add 27 is 147.

Ms. J: Guess, check and refine is a good way to solve problems. We're going to use it a lot this year. I'm putting it up on the wall so we'll all remember how important it is! Let's try another problem....

GUESS AND CHECK

WEEK 1 - DAY 1

a. Who am I? If you multiply me by 15 and then add 27, the result is 147.

b. Who am I? If you subtract me from 30 and then double that answer, the result is 48.

c. Who am I? I'm a 2-digit number. If you reverse my digits and add us together, the result is 110. (Find other correct answers.)

****

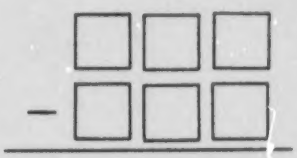
WEEK 1 - DAY 2

Julia and Ed were asked to add two numbers together. Julia, by mistake, subtracted the two numbers and gave the answer as 10. Instead of adding, Ed multiplied the two numbers and the result was 651. What was the correct total?

****

WEEK 1 - DAY 3

Use each of the digits 0, 1, 2, 3, 4, 5. Find the smallest 2-digit answer.



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WEEK 1 - DAY 4

The Happy Holiday Hotel is blessed with cheerful bedbugs. In each single bed you can find 7 bedbugs and in each double bed there are 13 bedbugs. If there are 106 bedbugs altogether, how many double beds are there?

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Guess And Check

Day 1. Answers:

a. 6 b. 6 c. 19 + 91 = 110
28 + 82 = 110
37 + 73 = 110
46 + 64 = 110
55 + 55 = 110

Comments and suggestions:

. This problem gives you an opportunity to demonstrate the "refinement" process. For example, in part a, first show that "5" is too small and "9" is too large. The answer must be between 5 and 9.

Day 2. Answer:

31 - 21 = 10
31 x 21 = 651

The correct total is 52 (31 + 21)

Comments and suggestions:

. A first guess might be 15 and 5 (a difference of 10). But the product of 15 and 5 is only 75 and a product of 651 is required.

. The next guess might be 30 and 20. The product is still too small, but much closer than before.

Day 3. Answer:

```
  3 0 1
- 2 5 4
-------
    4 7
```

Comments and suggestions:

. A common pupil response is

```
  5 3 1
- 4 2 0
-------
  1 1 1
```

You might explain how this guess can be refined by switching the ten's digits and also the units digits as shown.

```
  5 2 0
- 4 3 1
-------
    8 9
```

Day 4. Answer:

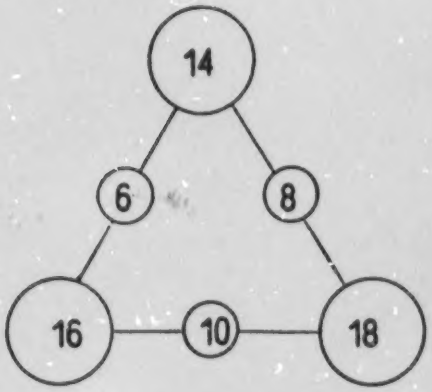
6 double beds

Comments and suggestions:

. First try 9 double beds. That's too many since 9 x 13 = 117. Therefore, the guesses (double beds) should be less than 9.

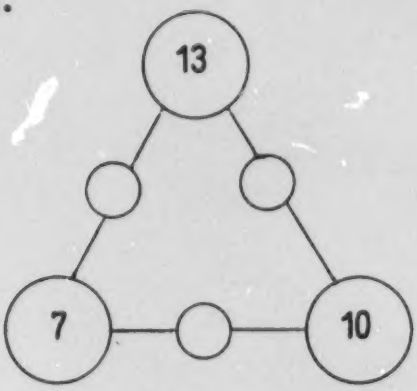
5. The numbers in the big circles are found by adding the numbers in the small circles.

Example:

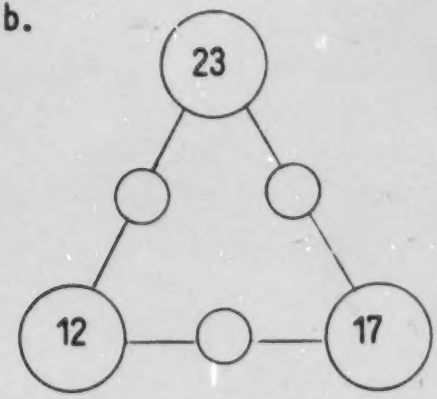


Find the numbers for the small circles in each problem.

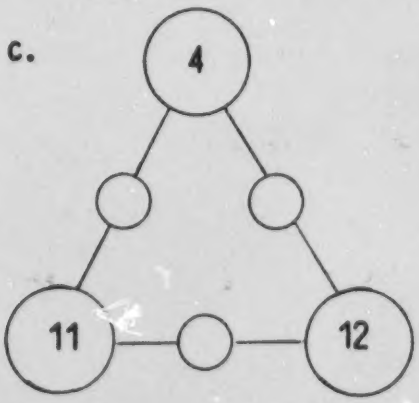
a.



b.

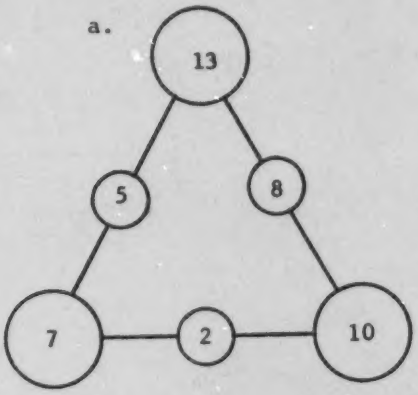


c.

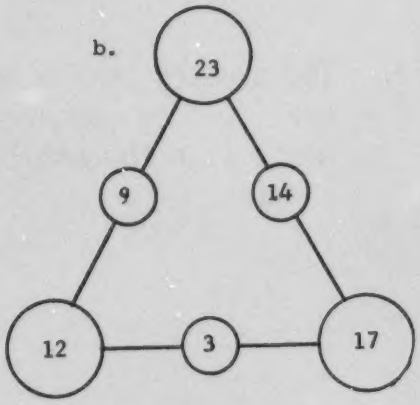


Day 5. Answers:

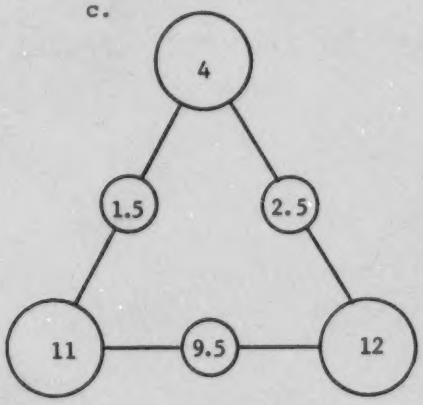
a.



b.



c.



Comments and suggestions:

- These problems provide many opportunities to use the "refinement" process.
- Pupils may think that part c is impossible until you suggest using decimals.

Look For A Pattern

By now your pupils are familiar with the skill guess and check. One teacher introduced the next problem-solving skill, look for a pattern, in this way.

Mr. Todd: Who remembers what method we used to solve problems last week?

Teri: We guessed.

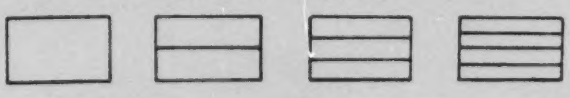
Mr. Todd: Is that all?

Sam: Guess and check. It's up on the poster!

Mr. Todd: That's right and we're going to add another problem-solving skill to the poster today. (Writes it up.) What does it say?

Class: Look for a pattern.

Mr. Todd: This week we are going to practice looking for patterns. Here's our problem. (Shows problem on overhead.) Do you see how to get three rectangles in the second figure?



Ellen: Yes. There are two skinny rectangles and one big one.

Mr. Todd: How can you get six rectangles in the third figure? Sid?

Sid: There are three skinny ones, two middle-sized ones, and one big one. A total of six.

Mr. Todd: How many rectangles are in the fourth drawing? Sue?

Sue: Four skinny ones; three fairly skinny ones, and two fatter ones--that's nine.

Sid: She forgot the large one. There should be ten altogether.

Mr. Todd: That's correct. Let's put our information in a table to see if there's any pattern. (Writes following table on chalkboard.)

Drawing	1st	2nd	3rd	4th	5th	6th	10th
Number of Rectangles	1	3	6	10			

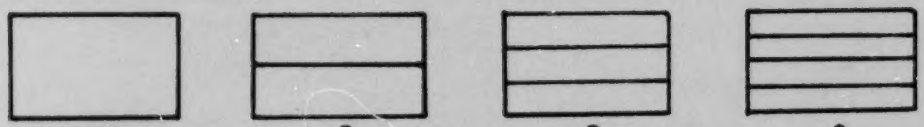
What kind of a pattern do you see?

Jim: It jumps by two, then by three, then by four, then by five. The next number should be fifteen.

Mr. Todd: Good. Perhaps some of you have discovered other ways to arrive at this pattern. (Other pupils suggest ways and Mr. Todd discusses each of them.)

Mr. Todd: By looking for patterns you've been able to solve a difficult problem. Patterns are very helpful in problem solving. Tomorrow we'll look at another problem that can be solved by patterns.

WEEK 2 - DAY 1



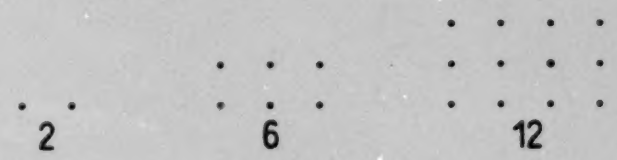
Notice that -
- . the 1st drawing contains 1 rectangle;
- . the 2nd drawing contains 3 rectangles;
- . the 3rd drawing contains 6 rectangles.

How many rectangles are in the 4th drawing?

If we continued the pattern, how many rectangles would be in
- . the 5th drawing?
- . the 6th drawing?
- . the 10th drawing?

****

WEEK 2 - DAY 2



The 1st figure contains 2 dots.
The 2nd figure contains 6 dots.
The 3rd figure contains 12 dots.

If you made a drawing of the 4th figure, how many dots would it contain? How many dots would the 10th figure contain?

WEEK 2 - DAY 3

Look For A Pattern

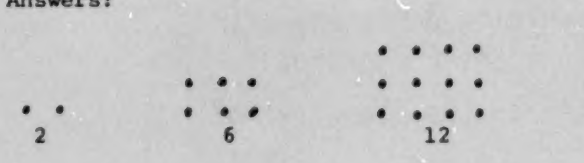
Day 1. Answers: The answers are shown in the table.

Drawing	1st	2nd	3rd	4th	5th ↓	6th ↓	7th	8th	9th	10th ↓
Rectangles	1	3	6	10	15	21	28	36	45	55

Comments and suggestions:

. A detailed description of how to present this problem is given in the introduction to this skill on page 11.

Day 2. Answers:



The 4th figure contains 20 dots. The 10th figure would contain 110 dots (10 x 11).

Comments and suggestions:

. Note the number pattern: the numbers first increase by 4, then by 6, then by 8, etc.

. Note also the dot pattern: the first array is 1 by 2, the second is a 2 by 3 ... the tenth is a 10 by 11.

Day 3. Answers:

a. + 5

8	3
15	10
30	25
50	45
25	20
105	100

b. x 4

5	20
6	24
10	40
8	32
25	100
15	60

c. x itself

1	1
2	4
3	9
4	16
10	100
9	81

d. x3, then +1

2	7
3	10
5	16
8	25
22	67
7	22

Comments and suggestions:

. It may be necessary to point out that the numbers given in the tables are not organized vertically in any way. Instead, the pupils need to look for a pattern that relates the first numbers with the second numbers in each table.

WEEK 2 - DAY 3

In each case, find the rule that will give the second number if you know the first. Then fill in the rest of the table according to the rule.

a.

8	3
15	10
30	25
50	?
25	?
?	100

b.

5	20
6	24
10	40
8	?
?	100
?	60

c.

1	1
2	4
3	9
4	?
10	?
?	81

d.

2	7
3	10
5	16
8	?
22	?
?	22

****

WEEK 2 - DAY 4

Have your next party at
SKAGGS SKATING

Prices - First 10 people $15.00
- Each additional person $ 1.25

Complete the table. Look for patterns.

People	10	11	12	13	14	15	16	17	18	22	26	30
Total Cost	$15	?	?	?	?	?	?	?	?	?	?	?

****

WEEK 2 --DAY 5

Ten people are at a party. Each person shakes hands with each of the others. What is the total number of handshakes?

Start with easier cases. Complete the table. Look for patterns.

People	2	3	4	5	6	7	8	9	10
Handshakes	1	3							

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Day 4. Answers:

People	10	11	12	13	14	15	16	17	18	22	26	30
Total Cost	$15	$16.25	$17.50	$18.75	$20	$21.25	$22.50	$23.75	$25	$30	$35	$40

Comments and suggestions:

. Two patterns should be discussed. First, as the number of people increases by 1, the cost increases by $1.25. Also, as the number of people increases by 4, the cost increases by $5.

Day 5. Answers:

People	2	3	4	5	6	7	8	9	10
Handshakes	1	3	6	10	15	21	28	36	45

Comments and suggestions:

. There are several methods that pupils might use to complete the pattern. Here are two of them.

- The number of handshakes first increases by 2, then by 3, then by 4, etc.
- To get the number of handshakes, add the numbers in the previous column of the table. For example, 10 handshakes (5 people) is equal to 4 + 6.

. This pattern is the same as the one in the "rectangles" problem (problem 1). In fact, this pattern shows up often in mathematical investigations.

Make A Systematic List

The first problem on the next page can be used to motivate the next problem-solving skill, make a systematic list. Mr. Benton showed the problem on an overhead.

Mr. B: What are the dimensions of some possible frames?

Al: 15 by 15; 20 by 15; 10 by 25 .

Mr. B: (Writes these results on the chalkboard.) Any others?

Barb: 10 by 10; 15 by 25; 25 by 25 .

Mr. B: (Adds these to his list.) Are there more?

Brian: I think so. 20 by 20 and 15 by 20 .

Joe: We already have the last one.

Mr. B: Perhaps if we make a systematic list we may not have the problem of duplicates. Let's first list all possibilities with 10 inches as one of the dimensions. (Shows on chalkboard.)

10 by 10
10 by 15
10 by 20
10 by 25

What would you suggest next? Julie?

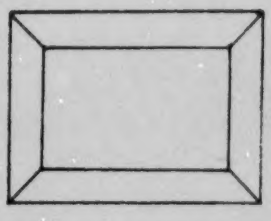
Julie: List the new frames which have 15 inches as one of the dimensions. Then those with 20, ...

Mr. B: (Completes the organized list showing all 10 possibilities.) Do you see how making a systematic list can help us solve a problem?

WEEK 3 - DAY 1

Mr. Young makes picture frames like the one illustrated. He makes them from pieces of wood with measurements of

10 inches, 15 inches, 20 inches, and 25 inches.



One possible frame is a 10 by 10; another is a 15 by 25.
List all the other possible frames.

****

WEEK 3 - DAY 2

Slim was studying the menu. He had the following choices.

1st course - Soup or salad
2nd course - Chicken, beef, lamb, or ham
3rd course - Pie, cake, or sherbert

List all possible combinations of three-course meals Slim could order.

****

WEEK 3 - DAY 3

There are 6 basketball teams in a tournament. The teams are lettered A through F. Each team plays each of the other teams twice. How many games are played altogether?

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Make A Systematic List

Day 1. Answers: There are ten possible frames.

10 by 10	15 by 15	20 by 20	25 by 25
10 by 15	15 by 20	20 by 25	
10 by 20	15 by 25		
10 by 25			

Comments and suggestions:

. Notice that in the first column all frames with a 10-inch dimension are listed; then those with a 15-inch dimension, etc. Also, by organizing the list in this way certain patterns show up.

Day 2. Answers: There are 24 possible combinations.

Soup-Chicken-Pie	Soup-Beef-Pie	Soup-Lamb-Pie	Soup-Ham-Pie
Soup-Chicken-Cake	Soup-Beef-Cake	Soup-Lamb-Cake	Soup-Ham-Cake
Soup-Chicken-Sherbert	Soup-Beef-Sherbert	Soup-Lamb-Sherbert	Soup-Ham-Sherbert

The other 12 combinations are the same as above except that "soup" is replaced by "salad."

Comments and suggestions:

. This might be a good time to show the beginning of a very disorganized list:

Soup-Chicken-Pie
Salad-Lamb-Sherbert
Salad-Beef-Cake
Soup-Ham-Pie

Hopefully, the class would soon recognize the importance of organization.

. You might suggest using symbols or abbreviations for the items to make the possibilities easier to list.

Day 3. Answers: 30 games (15 x 2)

AB				
AC	BC			
AD	BD	CD		
AE	BE	CE	DE	
AF	BF	CF	DF	EF

Comments and suggestions:

. Once again, certain patterns often show up when a list is organized. See if pupils can extend the pattern by asking how many games would be played if 7 teams were in the tournament.

20-

WEEK 3 - DAY 4

The corner store sells pencils for 5¢, 10¢, and 15¢.
List all the ways Kay can spend exactly 45¢ on pencils.

****

WEEK 3 - DAY 5

Mr. Moneybags has nickels, dimes, and quarters.
In how many ways can he make change for 50¢?

Day 4. Answers: 12 possibilities

15¢	10¢	5¢
3		
2	1	1
2		3
1	3	
1	2	2
1	1	4
1		6
	4	1
	3	3
	2	5
	1	7
		9

Comments and suggestions:

. Note that all combinations using at least one 15¢ pencil are systematically listed first. Then, the problem is reduced to a listing of just 10¢ and 5¢ pencils.

Day 5. Answers: Ten possible ways

25¢	10¢	5¢
2		
1	2	1
1	1	3
		5
	5	
	4	2
	3	4
	2	6
	1	8
		10

Comments and suggestions:

. As in the "Day 4" problem the list is organized by first considering all ways to use at least one 25¢ piece. Then the problem is reduced to a listing of just 10¢ and 5¢ coins.

Make And Use A Drawing Or Model

Usually the problem-solving skills are introduced directly, but sometimes they can be brought out by taking advantage of pupil efforts. This problem was given in the fourth week of problem solving in Ms. Thompson's class:

> A frog is at the bottom of a 10-foot well. Every day he crawls up 3 feet. But at night he slips down 2 feet. How many days will it take for him to get out of the well?

Pupils were allowed to think about the problem for a few minutes. Soon some thought they had solved it; others weren't sure and some had given up.

Ms. T: Nora, you wrote down some things to try. Will you share what you did?

Nora: Well, I guessed 7 days and counted up and down but it didn't work.

Ms. T: You used a guess and check approach. That's good but I also see you made a drawing on your paper.

Nora: I had to so I could keep track.

Ms. T: I saw several of you making a drawing on your paper. Do you know what our next problem-solving skill is going to be? Make and use a drawing or model. That sounds long and complicated but some of you are already doing it. Let's look at some of the drawings you tried. Tom, will you describe what you did?

Tom: I made a diagram and divided it into 10 parts. Then I actually moved up and down according to the "rules."

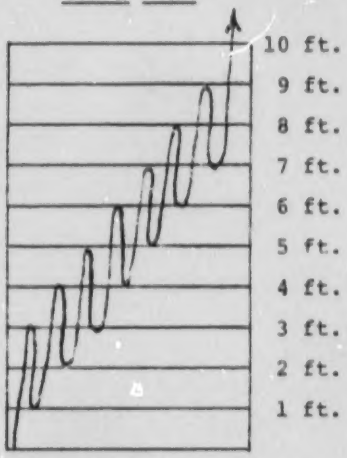
Ms. T: Will you show us?

Tom: Yes. I thought the answer would be 10 days. But as I move up and down like this, I find that at the end of the 7th day I only have 3 more feet to go.

Ms. T: We see that the diagram is very, very helpful. What is the answer to the problem?

Class: 8 days.

Tom's Work



WEEK 4 - DAY 1

A frog is at the bottom of a 10-foot well. Every day he crawls up 3 feet. But at night he slips down 2 feet. How many days will it take for him to get out of the well?

****

WEEK 4 - DAY 2

It takes 27 sugar cubes to fill a cubical box.

a. How many cubes will be touching the bottom of the box?

b. How many cubes will be touching the sides?

****

WEEK 4 - DAY 3

Six girls have just completed a 400-metre race. The girl's names are Chris, Debbi, Julie, Kay, Nancy, and Vicki. Use the following clues to help you determine in which order the girls finished.

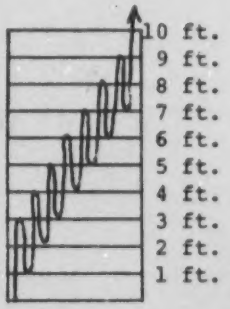
- Nancy beat Julie by 6 metres.
- Chris finished 11 metres behind Vicki.
- Nancy finished 2 metres ahead of Kay but 3 metres behind Vicki.
- Debbi finished exactly halfway between the first and last person.

Make And Use A Drawing Or Model

Day 1. Answer: 8 days

Comments and suggestions:

. You should make a drawing like the one pictured and show the frog's day-by-day "progress."

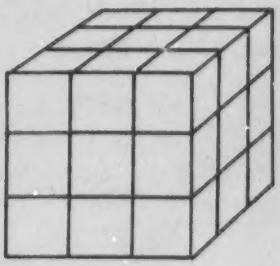


Day 2. Answers: a. 9 cubes b. 24 cubes

Comments and suggestions:

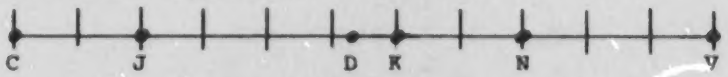
. Once a drawing is made, the problem becomes much easier to solve.

. If available, wooden cubes or sugar cubes also could be used.



Day 3. Answers: 1st: Vicki
2nd: Nancy
3rd: Kay
4th: Debbi
5th: Julie
6th: Chris

Comments and suggestions:



. This diagram is a scale drawing of the finish of the race. (1 cm = 1 m).

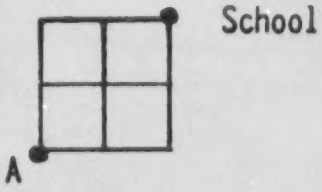
WEEK 4 - DAY 4

A service station has a large container of antifreeze. A customer needs exactly 2 quarts. The only containers available are a 5-quart can and a 4-quart can. How can the attendant use these containers to measure out the 2 quarts?

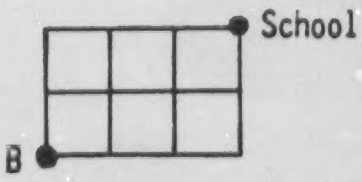
****

WEEK 4 - DAY 5

Ann lives at A. There are 6 different routes she can use to walk to school. Can you find them all? Each route should be 4 blocks?



Bill lives at B. How many different routes can he use to get to school? Each route should be 5 blocks.

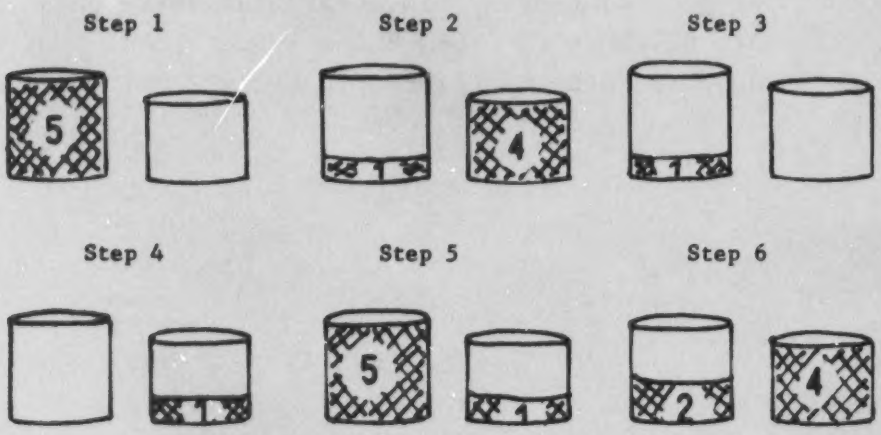


Day 4. Answer: This problem can be done in 6 steps.

Comments and suggestions:

. A series of drawings can be used to show the solution (see below).

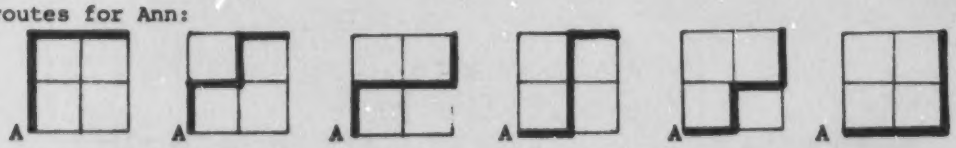
. Or you can draw the two containers on the chalkboard and use an eraser and chalk to show the change in levels at each stage.



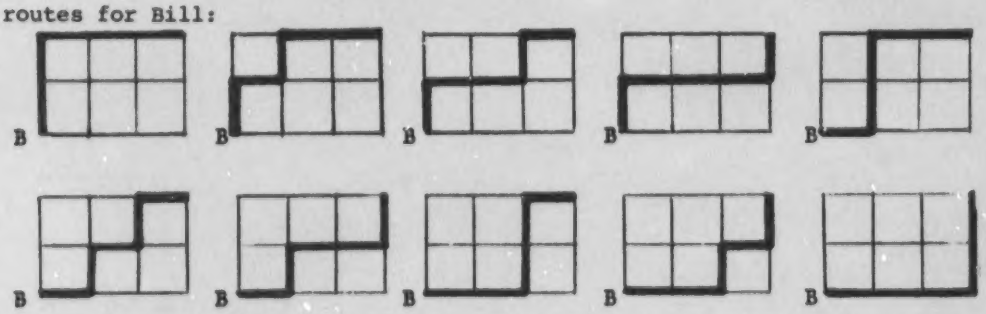
Of course, the easiest solution is to measure out one quart (as in Step 2), pour it into the car, and then repeat the process.

Day 5. Answers:

6 routes for Ann:



10 routes for Bill:



Comments and suggestions:

. The pupils should realize that if the diagrams are made with some system in mind, they will have more success in finding all possible routes.

-28-

Eliminate Possibilities

Usually pupils try to solve a problem by looking directly for the answers, but sometimes it is more helpful to identify or list possible answers, then eliminate incorrect answers. This narrows the search for a correct answer and in some cases the correct answer is all that is left.

Here is how Mr. Sopa introduced eliminate possibilities in the fifth week of problem solving.

Mr. S: I am thinking of a number between 1 and 100. Ask me questions to figure out the number. Word your questions so I can answer yes or no. Try to use as few questions as possible.

Mona: Is it greater than 50?

Mr. S: No. (Write on board >50? No)

Rudy: Is it less than 25?

Mr. S: No. (Writes <25? No)

Jane: Is it less than 15? No, wait--it can't be--I mean is it less than 35?

Mr. S: Yes. (Writes <35? Yes)

Ron: Is it 30?

Mr. S: No. (Writes 30? No)

Ron: I should have worded that differently. Now we don't know what side of 30 it's on! Is it less than 30?

Mr. S: Yes. (Writes <30? Yes)

Mike: Is it less than 28?

Mr. S: No. (Writes <28? No)

Kari: That means 28 or 29. Is it 29?

Mr. S: Yes--that's pretty good. You must have played this game before. Let's look at the strategy you used. Most of you thought about the possible numbers and then asked a question that eliminated half of the possibilities. We are going to concentrate on that problem-solving skill for this week--eliminate possibilities.

ELIMINATE POSSIBILITIES

WEEK 5 - DAY 1

Choose someone in the class to think of a number between 1 and 100. Ask this person questions that will allow you to eliminate possibilities...The questions should be worded so as to have a "yes" or "no" answer.

How many questions did it take the class to find the number? What are the fewest number of questions required?

Play this game several times.

****

WEEK 5 - DAY 2

In a certain bank the jobs of cashier, manager, and teller are held by Brown, Jones, and Smith but not necessarily in that order.

a. The teller, who was an only child, earns the least.

b. Smith, who married Brown's sister, earns more than the manager.

Who has which job?

****

WEEK 5 - DAY 3

Find four numbers from the box below whose sum is 200.

40	51	61	71	81	91
41	59	69	79	85	95
45				89	99
49					

Eliminate Possibilities

Day 1. Answers: Answers will vary.

Comments and suggestions:

. The most efficient strategy is to eliminate one half of the possible answers with each guess. Pupils can do this as described in the introductory commentary for this skill or by asking if the number is between 50 and 100, etc.

Day 2. Answer: Cashier is Smith, Manager is Brown, Teller is Jones.

Comments and suggestions:

. Without considering conditions a and b there are 9 possibilities. These can be shown with a grid.

. Certain possibilities can be eliminated and X'd out. For Example:

X_1: Brown has a sister so can't be the teller.

X_2: Smith earns more than some so can't be the teller who earns the least.

	Brown	Smith	Jones
Cashier		Yes	X_3
Manager	Yes	X_5	X_4
Teller	X_1	X_2	Yes

X_3 & X_4: Jones has to be the teller so can't be the cashier or manager.

X_5: Smith makes more than the manager so can't be the manager. Smith must be the cashier and Brown the manager.

Day 3. Answer: 41 + 49 + 51 + 59

Comments and suggestions:

. It is tempting to start looking for an answer directly by using guess and check. But this could be discouraging (there are 1820 combinations that would need to be tried). A more efficient way is to eliminate possibilities.

. Suppose we tried 99 as one of the numbers. The sum of the other three would have to be 101. But the total of the three smallest numbers is 126, so 99 is not possible. All the numbers above 71 can also be eliminated in this way.

. Notice that 40 cannot be used; it is the only even number and cannot be added to 3 odds to give 200.

. Also, 45 can be eliminated since 85 or 95 cannot be used.

. Thus, with these eliminations, the list becomes much smaller and the problem becomes easier to handle.

WEEK 5 - DAY 4

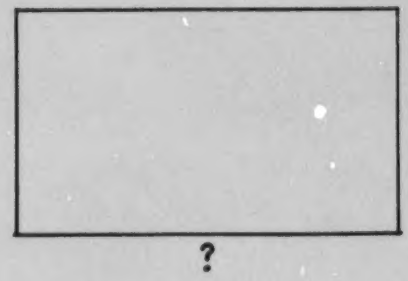
What number times itself is 4096?

****

WEEK 5 - DAY 5

The area of a rectangle is 209 square inches.

Find the length and width. (The length and width are whole numbers greater than 1.)



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Eliminate Possibilities

Day 4. Answer: 64

Comments and suggestions:

. Before any trials are made, all possibilities more than 70 can be eliminated since 70 x 70 = 4900.

. Also, since the units digit in the product is 6, all factors, except those ending in 4 or 6, can be eliminated.

Day 5. Answer: 11 x 19

Comments and suggestions:

. Ask pupils if it is possible for the length to be 2, or 4, or any even numbers. No. If either of the numbers were even, the area would be an even number. So, even numbers can be eliminated.

. What odd divisors are possible: 3, 5, 7, 9, 11, 13, 15? You can cross off 5 immediately because 209 doesn't end in 0 or 5. Using the divisibility rule for 3 helps eliminate 3 quickly. Try 7. No. 11? 11 x 19 = 209. Both 11 and 19 are prime so 11 x 19 is the only possible answer.

Grade 8

II. DRILL AND PRACTICE

II. DRILL AND PRACTICE

Most pupils in eighth grade classes have varying levels of skills. What can be done to provide additional practice and learning for all pupils? One solution is to offer drill and practice through problem-solving activities. While Tim is remembering that 8 x 7 is 56, not 54, Hosea might be figuring out all the possible ways to complete _____ x _____ = 48.

One marvelous discovery in using problem-solving skills with a mixed class is that some pupils who are unskilled in computation are really good at seeing patterns or figuring out ways to solve problems. Some who detest ordinary drill problems will fill a page with computation to try to solve a problem (see "Find A Path," page 57).

Using the activities

The activities in this Drill And Practice section can be incorporated with the regular teaching and review of whole numbers, fractions, and decimals. Most of them should be done in class so that the teacher can interact and give appropriate hints when necessary. There are many opportunities for pupils to create problems of their own. This type of activity should be encouraged since pupils will become more aware of the real structure of a problem if they have a chance to invent one of their own.

The three lessons entitled "Best Guess" are intended to develop pupils' estimation skills. It is important that these lessons be done prior to the lessons in Unit VIII - Percent Estimation.

COMPLETE THE PROBLEM

1. $2345 + \square\square\square\square = 6789$

2. $1579 + \square\square\square\square = 10000$

3. $4671 - \square\square\square\square = 2888$

4. $\square 1 \square 2 - 7 \square 9 \square = 1111$

5. $\square 53\square - 948 = \square\square 4$

6. $\square\square\square \times 7 = 2345$

7. $123 \times \square\square = \square\square 5 + \square\square\square = 553\square$

8. $8\square\square \times \square 7 = \square\square 63 + \square\square 6\square = \square 2\square 9\square$

9. $26\square \times 4\square = 134\square + \square 0\square\square = 1\square 060$

10. $\square\square\square \div 24 = 26$

11. $\square \div 216 = 27$

12. $\square 9 \div 73 = 2\,1\,1\square$

$1\,4\,6$

$\square\square\square$

$\square\square\square$

2

13. $\square\square 2971 \div \square\square = 5\square$

265

$\square\square\square$

$\square\square\square$

3

14. $215 \div \square\square\square = 1\square\square$

$\square\square\square$

$\square 5\square 9$

$\square 5\square 5$

$\square 4\square$

$\square\square\square$

0

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Complete The Problem

Mathematics teaching objectives:

- Reinforce whole number computation skills.
- Use inverse operations.

Problem-solving skills pupils might use:

- Guess and check.
- Recognize limits and eliminate possibilities.
- Work backwards (inverse operations).

Materials needed:

- None

Comments and suggestions:

- It is best if pupils work independently on these problems.
- Allow ample time in class so that pupils can ask questions and you can give appropriate hints when necessary. In fact, this may be best as a two-day lesson.
- If pupils get finished early, have them create a similar problem of their own.

Answers:

1. 2345 + 4444 = 6789

2. 1579 + 8421 = 10000

3. 4671 - 1783 = 2888

4. 9102 - 7991 = 1111

5. 1532 - 948 = 584

6. 335 x 7 = 2345

7. 123 x 45 = 615 + 492 = 5535

8. 809 x 77 = 5663 + 5663 = 62293

9. 268 x 45 = 1340 + 10720 = 12060

10. 24 | 624 = 26

11. 27 | 216 = 8

12. 73 | 2119 = 29 R 2 (146 + 657)

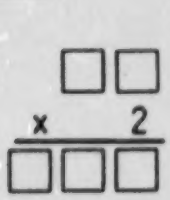
13. 53 | 2971 = 56 R 3 (265 + 318)

14. 215 | 37195 = 173 (215 + 1569 + 1505 + 645 + 645)

MORE THAN ONE SOLUTION

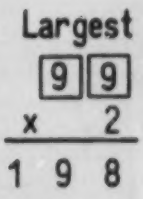
The problems below have more than one solution. In each case find the smallest possible solution and the largest possible solution.

Example

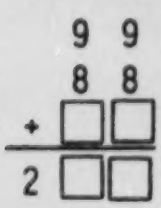


Smallest

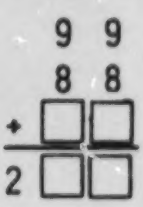
Largest



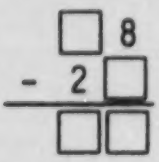
1. Smallest



Largest



2. Smallest



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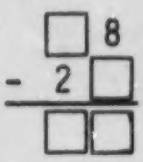
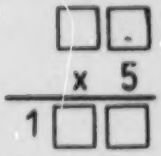


3. Smallest



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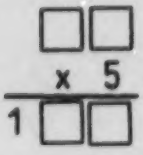
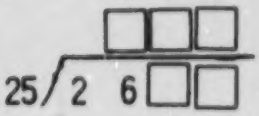


4. Smallest



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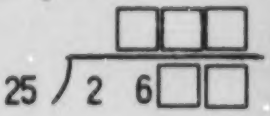
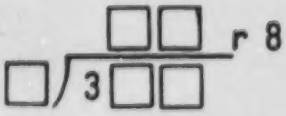
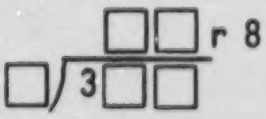


5. Smallest



Largest



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More Than One Solution

Mathematics teaching objectives:

- Reinforce whole number computation skills.
- Use inverse operations.

Problem-solving skills pupils might use:

- Guess and check.
- Recognize limits and eliminate possibilities.
- Work backwards.

Materials needed:

- None

Comments and suggestions:

- It is best if pupils work independently on these problems.
- Allow ample time in class so that pupils can ask questions and you can give appropriate hints when necessary.
- If pupils get finished early, have them create a similar problem of their own.
- When lesson is completed, have pupils discuss the ways in which they arrived at the limits to the answers.
- If zeros are allowed, 2 (smallest) might be either $\boxed{2}8 - 2\boxed{8} = \boxed{0}\boxed{0}$ or $\boxed{3}8 - 2\boxed{9} = \boxed{0}\boxed{9}$.

Answers:

1.

Smallest	Largest
9	99
88	88
+13	+99
200	286

2.

Smallest	Largest
38	98
- 28	- 20
10	78

3.

Smallest	Largest
20	39
x5	x5
100	195

The answer must be a multiple of 5.

4.

Smallest	Largest
$25\overline{)2600}$	$25\overline{)2675}$
104	107

5.

Smallest	Largest
$9\overline{)305}$	$9\overline{)395}$
33 r8	43 r8

Note: A remainder of 8 implies that the single digit divisor must be 9.

DECIMAL DIGIT DRAW ACTIVITIES
(Ideas for Teachers)

Ten digit cards marked 0-9 can be used for a variety of activities.

A. Have pupils draw the diagram to the right.

B. Suggest a goal, such as getting the largest sum.

C. Draw digits, one at a time, until all spaces are filled.

D. Pupils must write a digit each time and no changes can be made.

E. Compare and discuss results.

This activity is very adaptable. For example, in the 3-digit by 3-digit addition problem above, the teacher might have the pupils use the six digits to first find the largest sum; then use the same six digits to find the smallest sum; and then use those digits to find the sum closest to, say 70.

These activities can be used for drill and practice, concept development and/or diagnosis. Each is highly individual, as many pupils will have a unique problem. See the next page for other suggested formats. Further variations could include:

- replacing a drawn digit so it can be used again
- restricting certain numbers, e.g., no zero is used with division problems
- allowing a special reject box giving pupils the chance to discard an unfavorable draw.

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Decimal Digit Draw Activities

Mathematics teaching objectives:

- Practice computation skills.
- Develop place value concepts.
- Develop informal probability concepts.

Problem-solving skills pupils might use:

- Break a problem into manageable parts.
- Make decisions based upon data.
- Recognize limits and/or eliminate possibilities.

Materials needed:

- Digit cards 0-9

Comments and suggestions:

- This activity works well as an opener at the beginning of the class or as an ending for that last few minutes of a class when all other activities have been completed. See pages 43 and 45 for other suggestions and many variations.
- After some trials, pupils can be encouraged to share the strategies they use. "If a 9 or 8 is drawn first, put it in the 100's place. If a little digit comes first, put it in the 1's place. It's hard to decide what to do with a 5 or 6. Sometimes it's just luck!"

Decimal Digit Draw Activities (cont.)

Suggested formats:

Place Value: .□□□ largest number, smallest number, or closest to .5

Ordering: .□□□ < .□□ < .□

Addition: □□ + .□□ + □.□

□.□ + .□□

Subtraction: □□.□ − □.□□

□ − □.□□

Multiplication: □.□ × □.□ closest to 20

.□□ × .□□□ smallest number

Division: .□□□ ÷ .□

DISTRIBUTIVE PROPERTY

1. Match each problem with the correct answer. Several answers are used more than once.

Problems	Answers
a. $9(14 + 16) =$	A. 520
b. $(9 \times 14) + (9 \times 16) =$	B. 375
c. $(25 \times 4) + (25 \times 11) =$	C. 270
d. $25(4 + 11) =$	D. 100
e. $13(18 + 22) =$	E. Answer not given
f. $(13 \times 18) + (13 \times 22) =$	
g. $(35 \times 12) + (35 \times 88) =$	
h. $35(12 + 88) =$	

2. Match the problems which have the same answer. Not all of them can be matched up.

a. $3(5 + 17)$	A. $5(4 + 8)$
b. $10(8 + 4)$	B. $(150 \times 29) + (150 \times 62)$
c. $(5 \times 4) + (5 \times 8)$	C. $7(3 + 10)$
d. $497(268 + 574)$	D. $(3 \times 5) + (3 \times 17)$
e. $(7 \times 3) + (7 \times 10)$	E. $9(6 + 4)$
f. $(9 \times 6) + (9 \times 5)$	F. $(10 \times 8) + (10 \times 4)$
g. $150(29 + 62)$	G. $(497 \times 268) + (497 \times 774)$

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Distributive Property

Mathematics teaching objectives:

- Discover and use the distributive property.
- Reinforce computation skills (mental as well as physical).

Problem-solving skills pupils might use:

- Change a problem into one that is easier to solve.
- Look for patterns.
- Create new problems similar to ones previously solved.

Materials needed:

- None

Comments and suggestions:

- Review what is meant by 9(14 + 16).
- It is important to have pupils share their discoveries in problems 1 and 2 before they proceed to problem 3.
- Most of the answers in problem 3 can be arrived at mentally by applying the distributive property. You may want to go through a couple of the problems with the class to get them started.
- Procedures for solving problems 4 and 5 need to be discussed.
- You may wish to provide the class with additional problems which use the distributive property and involve fractions and decimals.

Answers:

1. a. C
   b. C
   c. B
   d. B
   e. A
   f. A
   g. E
   h. E

2. a. D
   b. F
   c. A
   d. No "answer" given
   e. C
   f. No "answer" given
   g. B

3. The following exercises all can be done mentally. Find the answer to each.

   a. $(15 \times 95) + (15 \times 5)$

   b. $(6 \times 42) + (6 \times 8)$

   c. $5(11 + 20)$

   d. $4(\$.25 + \$9.00)$

   e. $(16 \times 991) + (16 \times 9)$

   f. $9(60 + 40)$

   g. $(\$2.85 \times 7) + (\$2.85 \times 3)$

   h. $(8 \times 2.6) + (8 \times 2.4)$

   i. $\frac{3}{2}\left(\frac{2}{3} + 2\right)$

4. Stacy bought 7 used records on Monday for 98¢ each. On Tuesday she bought 3 more for 98¢ each. How much did she spend altogether?

5. On Friday, Ronnie sold 65 tickets to the basketball game for $2.75 each. On Saturday he sold 35 more for $2.75 each. How much did he collect altogether?

Answers:

3. a. 1500
   b. 300
   c. 155
   d. $37
   e. 16,000
   f. 900 (This is easier if left in the original form.)
   g. $28.50
   h. 40
   i. 4

4. Can be done mentally by using distributive property. $.98(7 + 3) = $9.80

5. Can be done mentally by using distributive property. $2.75(65 + 35) = $275

6. a. (7 x 10) + (7 x 8)
   b. (5 x 4) + (5 x 16)
   c. (10 x 6) + (10 x 4)
   d. (25 x 8) + (25 x 2)

7. a. They are correct.
   b. Answers will vary.

8. Answers will vary. Two possibilities are given.

   $5(6 + 2) = 40$ $\qquad$ $4(8 + 2) = 40$

   $(5 \times 6) + (5 \times 2) = 40$ $\qquad$ $(4 \times 8) + (4 \times 2) = 40$

6. The property you've been using is called the Distributive Property. Here's an example:

$$5(8 + 2) = 50$$
$$(5 \times 8) + (5 \times 2) = 50$$

Use the distributive property to rewrite each of the following in another way.

a. $7(10 + 8)$

b. $5(4 + 16)$

c. $10(6 + 4)$

d. $25(8 + 2)$

7. Mr. Decker wanted his class to invent some expressions which showed the distributive property. But he wanted the answer to each of them to be 30.

a. Here are some of their inventions. Check to see that they are correct.

$$2(10 + 5) = 30$$
$$(2 \times 10) + (2 \times 5) = 30$$

$$3(6 + 4) = 30$$
$$(3 \times 6) + (3 \times 4) = 30$$

$$5(4 + 2) = 30$$
$$(5 \times 4) + (5 \times 2) = 30$$

b. Create another pair of expressions that show the distributive property. Your answers must still be 30.

8. Create 4 pairs of expressions that show the distributive property. The answer to each of them must be 40.

CREATE A PROBLEM

There are many problems that have an answer of 999.

An example using each of the four operations is given.

$$563 + 436 = 999$$
$$2146 - 1047 = 999$$
$$27 \times 37 = 999$$
$$14985 \div 15 = 999$$

Each number below is the answer to many problems. Make up four problems which have the given number as the answer. In each case, use four different operations.

1. 477
2. 12.8
3. .028
4. $\frac{15}{16}$
5. $3\frac{3}{4}$

EXTENSION

What five numbers have an average of 30.16?

Create A Problem

Mathematics teaching objectives:

- Reinforce computation skills (whole numbers, fractions, decimals).
- Use inverse operation.

Problem-solving skills pupils might use:

- Work backwards.
- Recognize limits.
- Search for or be aware of other solutions.

Materials needed:

- Calculators (optional)

Comments and suggestions:

- It is best to use this lesson after whole numbers, fractions, and decimals have been reviewed.
- Suggest to pupils that they not create "easy" problems like 476 + 1 = 477.
- Some pupils need to be reminded that these two forms of division are equivalent: $27 \div 3$ $\quad 3\overline{)27}$
- Allow ample time in class so that pupils can ask questions and you can give appropriate hints when necessary.
- You may want to suggest that pupils examine the problems using this kind of format:

$$\begin{array}{r} \square\square\square \\ +\ \square\square\square \\ \hline 4\ 7\ 7 \end{array} \qquad \begin{array}{r} \square\square \\ \times\ \ \square \\ \hline 4\ 7\ 7 \end{array}$$

- Pupils could be allowed to use calculators for the first three problems and the extension.
- A bulletin board display could be made showing some of the problems pupils have created.

Answers:

Answers will vary. Possible problems for numbers 3 and 4 are given.

3. .016 + .012 = .028
   .1 - .072 = .028
   .4 x .07 = .028
   .0056 ÷ .2 = .028

4. $\frac{1}{4} + \frac{11}{16} = \frac{15}{16}$

   $1 - \frac{1}{16} = \frac{15}{16}$

   $\frac{3}{2} \times \frac{5}{8} = \frac{15}{16}$

   $\frac{5}{8} \div \frac{2}{3} = \frac{15}{16}$

Extension: Answers will vary. The total, before dividing by 5, has to be 150.80 (5 x 30.16). One possible set of five numbers that have a sum of 150.80 is 10, 50, 40, 45.2, and 5.6.

PATTERNS

1. 5, 12, 19, 26, ___, ___, ___, ___, 61

2. 3, 4, 6, 9, 13, ___, ___, ___, 39

3. .5, .6, .7, .8, ___, ___, ___, ___, 1.3

4. .75, .8, .85, .9, ___, ___, ___, ___, 1.15

5. $1\frac{2}{5}$, $2\frac{4}{5}$, $4\frac{1}{5}$, $5\frac{3}{5}$, ___, ___, ___, ___, $12\frac{3}{5}$

6. 1, 4, 9, 16, 25, ___, ___, ___, 81

7. 2.14, 2.11, 2.08, 2.05, ___, ___, ___, ___, 1.9

8. $10\frac{1}{2}$, $9\frac{3}{4}$, 9, $8\frac{1}{4}$, ___, ___, ___, ___, $4\frac{1}{2}$

9. 5.52, ___, 5.3, ___, 5.08, ___, ___, ___, 4.64

10. .5, ___, $\frac{3}{2}$, ___, $2\frac{1}{2}$, ___, 3.5, ___, $\frac{9}{2}$

11. .077, ___, ___, ___, ___, ___, .131, .14, .149

12. 3, ___, 8, ___, 13, ___, ___, ___, 23

13. .2, .7, .65, 1.15, 1.1, ___, ___, ___, 2

EXTENSION

Create some patterns of your own. Leave some blanks. Give the problems to someone else to complete.

Patterns

Mathematics teaching objectives:

- Reinforce computation skills (mental as well as written).

Problem-solving skills pupils might use:

- Look for patterns.
- Guess and check.
- Detect and correct errors.
- Create new problems similar to ones already solved.

Materials needed:

- None

Comments and suggestions:

- It is best if pupils work independently on these problems.
- The pupils should recognize that the last number in each pattern can be used as a check. This will make it possible for them to detect, and perhaps correct errors.

Answers:

Pupils may find different patterns than those given below. These should be shared with the class.

1. 5, 12, 19, 26, 33, 40, 47, 54, 61
2. 3, 4, 6, 9, 13, 18, 24, 31, 39
3. .5, .6, .7, .8, .9, 1.0, 1.1, 1.2, 1.3
4. .75, .8, .85, .9, .95, 1.00, 1.05, 1.10, 1.15
5. $1\frac{2}{5}$, $2\frac{4}{5}$, $4\frac{1}{5}$, $5\frac{3}{5}$, 7, $8\frac{2}{5}$, $9\frac{4}{5}$, $11\frac{1}{5}$, $12\frac{3}{5}$
6. 1, 4, 9, 16, 25, 36, 49, 64, 81
7. 2.14, 2.11, 2.08, 2.05, 2.02, 1.99, 1.96, 1.93, 1.9
8. $10\frac{1}{2}$, $9\frac{3}{4}$, 9, $8\frac{1}{4}$, $7\frac{1}{2}$, $6\frac{3}{4}$, 6, $5\frac{1}{4}$, $4\frac{1}{2}$
9. 5.52, 5.41, 5.3, 5.19, 5.08, 4.97, 4.86, 4.75, 4.64
10. .5, 1, $\frac{3}{2}$, 2, $2\frac{1}{2}$, 3, 3.5, 4, $\frac{9}{2}$
11. .077, .086, .095, .104, .113, .122, .131, .14, .149
12. 3, 5.5, 8, 10.5, 13, 15.5, 18, 20.5, 23
13. .2, .7, .65, 1.15, 1.1, 1.6, 1.55, 2.05, 2

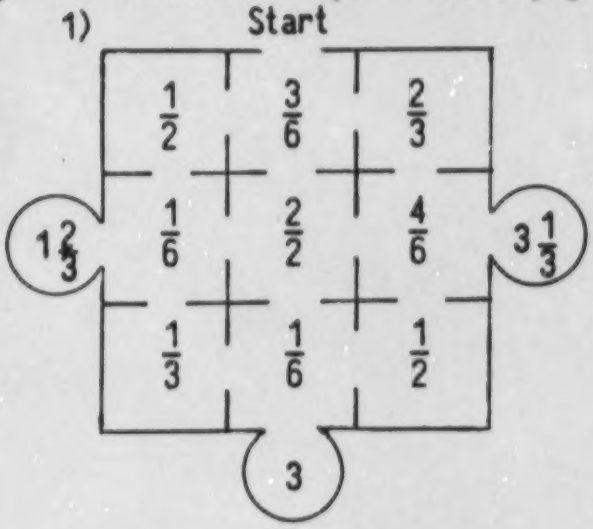
Extensions:

Answers will vary.

FIND A PATH

Draw a path so the numbers along the path add to the number in the circle. The path can only go through the open gates.

1)



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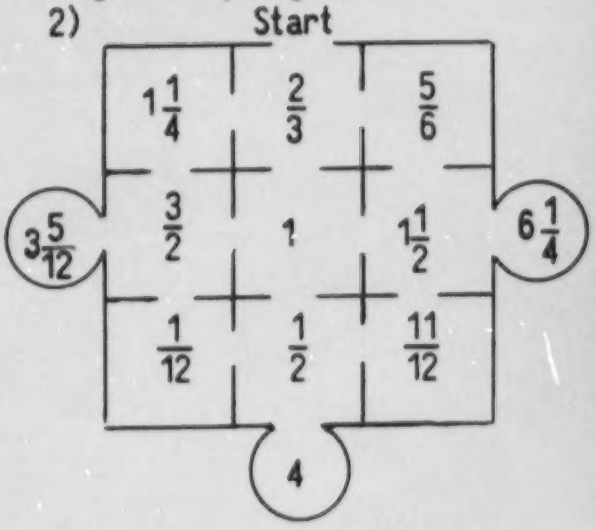

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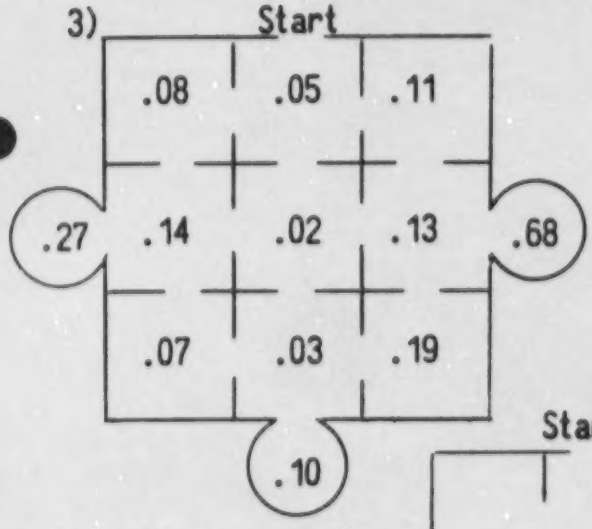
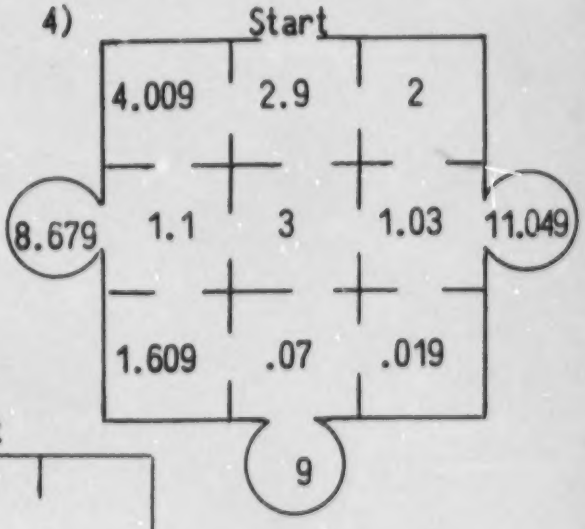
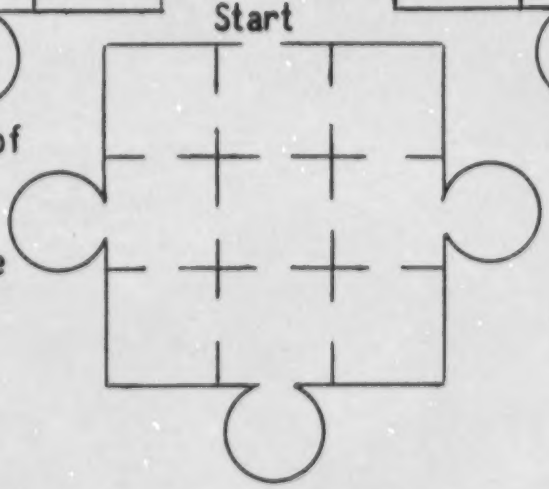


4)



5) Create one of your own. Give it to someone else to solve.



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Find A Path

Mathematics teaching objectives:

- Add and subtract fractions and decimals.

Problem-solving skills pupils might use:

- Recognize limits and eliminate possibilities.
- Guess and check.
- Solve a simpler problem.

Materials needed:

- None.

Comments and suggestions:

- You may need to guide the pupils through part of the first problem in order to get them started.
- The easiest problem in each box is the "left-circle" problem. The hardest is the "right-circle" problem. You may suggest that pupils work them according to their difficulty.
- It often is difficult to know what kinds of hints to give and when they should be given, if at all. For example, one could suggest changing the fractions to common denominators, but this may take away much of the problem. However, some pupils may need this kind of hint fairly early to keep them from becoming overly frustrated.
- Each problem can be simplified by noting that two squares must always be used--the first and the last. Thus, the sum of these could be subtracted from the total which results in an easier problem to solve.
- In problem 2 you must go through the middle square twice for the result to be $6\frac{1}{4}$. Note that the directions do not prohibit this. This also is true in problem 4. The middle and last square are each passed through twice. This "right circle" problem 4 is very difficult.

Answers:

1. $1\frac{2}{3} = \frac{3}{6} + \frac{2}{2} + \frac{1}{6}$

   $3 = \frac{3}{6} + \frac{2}{3} + \frac{4}{6} + \frac{2}{2} + \frac{1}{6}$

   $3\frac{1}{3} = \frac{3}{6} + \frac{1}{2} + \frac{1}{6} + \frac{1}{3} + \frac{1}{6} + \frac{2}{2} + \frac{4}{6}$

2. $3\frac{5}{12} = \frac{2}{3} + 1\frac{1}{4} + \frac{3}{2}$

   $4 = \frac{2}{3} + 1\frac{1}{4} + \frac{3}{2} + \frac{1}{12} + \frac{1}{2}$

   $5\frac{1}{4} = \frac{2}{3} + 1 + \frac{3}{2} + \frac{1}{12} + \frac{1}{2} + 1 + 1\frac{1}{2}$

3. $.27 = .05 + .08 + .14$
   $.10 = .05 + .02 + .03$
   $.68 = .05 + .08 + .14 + .07 + .03 + .02 + .05 + .11 + .13$

4. $8.679 = 2.9 + 3 + .07 + 1.609 + 1.1$
   $9 = 2.9 + 2 + 1.03 + 3 + .07$
   $11.049 = 2.9 + 3 + 1.03 + .019 + .07 + 3 + 1.03$

5. Answers will vary.

THE BEST GUESS
(MONEY)

Which answer is best?

1. $15.16 + $4.99

Less than	More than
$20	$20

2. $16.25 - $6.97

Less than	More than
$10	$10

3. 8 x $10.98

Less than	More than
$88	$88

4. $3.89 ÷ 4

Less than	More than
$1.00	$1.00

5. $2.94 + $3.15

Less than	More than
$6.00	$6.00

6. $20 - $9.89

Less than	More than
$10	$10

7. 12 x $4.75

Less than	More than
$50	$50

8. $8.85 ÷ 9

Less than	More than
$1.00	$1.00

9. $1.63 + $2.75 + $3.98

Less than	More than
$8.00	$8.00

10. $17.17 - $10.27

Less than	More than
$7.00	$7.00

11. 16 x $1.51

Less than	More than
$24	$24

12. $31.17 ÷ 8

Less than	More than
$3.50	$3.50

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The Best Guess (Money)

Mathematics teaching objectives:

- Estimate answers to problems involving money.
- Use rounding to help make estimates.

Problem-solving skills pupils might use:

- Make reasonable estimates.
- Recognize limits and eliminate possibilities.

Materials needed:

- Transparencies of each pupil sheet

Comments and suggestions:

- These estimation problems could be effectively used as a beginning of the period activity. Perhaps 12 problems are the most that should be given in any one session.

- Estimation should be an important part of the mathematics curriculum. Some authorities say that 75% of the adult non-occupational uses of math involve mental arithmetic and estimations. Frequently it is not necessary to have an exact answer, amount, or measurement. Pupils need many exposures to the estimation process. Some are reluctant to try. They would rather work a somewhat complicated problem in order to get "the" exact answer. The process needs to be taught (and practiced) using such individual skills as computing with single digits, rounding, and performing operations with powers and multiples of ten.

- This is a teacher-directed activity. Make overhead transparencies for each sheet. Project the problems on the screen but show only one at a time. Allow pupils a limited amount of time to respond--then give the answer, discuss if necessary and go on to another problem.

- You might want pupils to try to write down their answer before you give the correct answer. To motivate this, you could call it a "Beat the Teacher" activity. It is important, however, for pupils to use pencil and paper only to give their answer, not to do any computation whatsoever.

Answers:

1. More than $20.00
2. Less than $10.00
3. Less than $88.00
4. Less than $1.00
5. More than $6.00
6. More than $10.00
7. More than $50.00
8. Less than $1.00
9. More than $8.00
10. Less than $7.00
11. More than $24.00
12. More than $3.50
13. Between $38.00 and $39.00
14. Between $19.00 and $19.25
15. Between $15.00 and $18.00
16. Between $11.00 and $12.00
17. Between $110.00 and $120.00
18. Between $65.00 and $66.00
19. Between $100.00 and $101.00
20. Between $7.00 and $7.50
21. Between $2.80 and $2.90
22. Between $70.00 and $71.00
23. Between $35.00 and $40.00
24. Between $4.75 and $5.00

Some possible responses:

25. About $30.00
26. About $2.50
27. A little less than $24.00
28. About $15.00
29. Between $11.00 and $12.00
30. About $11.00
31. Between $100.00 and $110.00
32. A little more than $10.00
33. Between $5.00 and $6.00
34. A little more than $60.00
35. About $1400.00
36. A little less than $3.00

The Best Guess (Money) (cont.)

13. $12.91 + $13.84 + $11.99

Between $ 36 and $ 37
$ 37 and $ 38
$ 38 and $ 39

14. $25.01 - $5.95

Between $19.00 and $19.25
$19.25 and $19.50
$19.50 and $19.75

15. 9 x $1.99

Between $ 9 and $12
$12 and $15
$15 and $18

16. $100 ÷ 9

Between $ 10 and $ 11
$ 11 and $ 12
$ 12 and $ 13

17. $62.19 + $53.37

Between $100 and $110
$110 and $120
$120 and $130

18. $75.98 - $10.17

Between $ 64 and $ 65
$ 65 and $ 66
$ 66 and $ 67

19. 25 x $4.02

Between $100 and $101
$101 and $102
$102 and $103

20. $35.99 ÷ 5

Between $6.50 and $7.00
$7.00 and $7.50
$7.50 and $8.00

21. $.39 + $.89 + $.59 + $.99

Between $2.70 and $2.80
$2.80 and $2.90
$2.90 and $3.00

22. $100 - $29.84

Between $ 69 and $ 70
$ 70 and $ 71
$ 71 and $ 72

23. 15 x $2.49

Between $ 30 and $ 35
$ 35 and $ 40
$ 40 and $ 45

24. $14.85 ÷ 3

Between $4.75 and $5.00
$5.00 and $5.25
$5.25 and $5.50

The Best Guess (Money) (cont.)

Estimate the answer to each exercise. It may help to first determine limits within which the answer lies.

25. \$15.73 + \$14.09

26. \$4.56 - \$1.98

27. 8 x \$2.93

28. \$29.97 ÷ 2

29. \$6.96 + \$2.76 + \$1.75

30. \$16.91 - \$5.99

31. 11 x \$9.89

32. \$51.27 ÷ 5

33. \$.82 + \$.19 + \$1.23 + \$2.97

34. \$100 - \$39.19

35. 98 x \$13.79

36. \$35.71 ÷ 12

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THE BEST GUESS
(Decimals)

Which answer is best?

1. 2.93 + 5.78

Less than	More than
8	8

2. 16.6 - 4.9

Less than	More than
11	11

3. 1.9 x 3.9

Less than	More than
8	8

4. 84.9 ÷ 4.1

Less than	More than
20	20

5. 15.9 + 6.72

Less than	More than
22	22

6. 2.9 - 1.13

Less than	More than
2	2

7. 8.91 x .9

Less than	More than
10	10

8. 15 ÷ 1.9

Less than	More than
7	7

9. 15 + .9 + 6.92

Less than	More than
22	22

10. 53 - 2.017

Less than	More than
51	51

11. .47 x 23.9

Less than	More than
12	12

12. 12.1 ÷ .91

Less than	More than
12	12

The Best Guess (Decimals)

Mathematics teaching objectives:

- Estimate answers to problems involving decimals.
- Use rounding to help make estimates.

Problem-solving skills pupils might use:

- Make reasonable estimates.
- Recognize limits and eliminate possibilities.

Materials needed:

- Overhead transparencies for each lesson sheet

Comments and suggestions:

- These estimation problems could be used as a beginning of the period activity. Perhaps 12 problems are the most that should be given in any one session.
- Estimation should be an important part of the mathematics curriculum. Some authorities say that 75% of the adult non-occupational uses of math involve mental arithmetic and estimation. Frequently it is not necessary to have an exact answer, amount, or measurement. Pupils need many exposures to the estimation process. Some are reluctant to try. They would rather work a somewhat complicated problem in order to get "the" exact answer. The process needs to be taught (and practiced) using such individual skills as computing with single digits, rounding, and performing operations with powers and multiples of ten.
- This is a teacher-directed activity. Make overhead transparencies for each sheet. Project the problems on the screen but show only one at a time. Allow pupils a limited amount of time to respond--then give the answer, discuss if necessary and go on to another problem.
- You might want pupils to try to write down their answers before you give the correct answer. To motivate this, you could call it a "Beat the Teacher" activity. It is important, however, for pupils to use pencil and paper only to give their answers, not to do any computation whatsoever.

Answers:

1. More than 8
2. More than 11
3. Less than 8
4. More than 20
5. More than 22
6. Less than 2
7. Less than 10
8. More than 7
9. More than 22
10. Less than 51
11. Less than 12
12. More than 12
13. Between 17 and 18
14. Between 9 and 10
15. Between 40 and 45
16. Between 1 and 2
17. Between 41 and 42
18. Between 13 and 14
19. Between 14 and 17
20. Between 5 and 6
21. Between 32 and 33
22. Between 19 and 20
23. Between 0 and 1
24. Between 40 and 50

Some possible responses:

25. About 22
26. About 23
27. A little less than 4
28. About 10
29. Between 19 and 20
30. A little more than 7.4
31. A little more than 28
32. About 1
33. Between 38 and 39
34. A little more than 34
35. About 1
36. About .5

The Best Guess (Decimals) (cont.)

13. 12.68 + 1.29 + 3.59

Between 16 and 17
17 and 18
18 and 19

14. 29.7 - 19.9

Between 9 and 10
10 and 11
11 and 12

15. 5.99 x 6.9

Between 30 and 35
35 and 40
40 and 45

16. 16.33 ÷ 16.01

Between 0 and 1
1 and 2
2 and 3

17. 37.5 + 3.96

Between 40 and 41
41 and 42
42 and 43

18. 19.91 - 6.6

Between 13 and 14
14 and 15
15 and 16

19. .9 x 17.01

Between 14 and 17
17 and 20
20 and 23

20. 10.89 ÷ 2.1

Between 5 and 6
6 and 7
7 and 8

21. 29 + 2.9 + .29

Between 31 and 32
32 and 33
33 and 34

22. 27 - 7.93

Between 19 and 20
20 and 21
21 and 22

23. .091 x .13

Between 0 and 1
1 and 2
2 and 3

24. 100.1 ÷ 2.21

Between 30 and 40
40 and 50
50 and 60

The Best Guess (Decimals) (cont.)

Estimate the answer to each exercise. It may help to first determine limits within which the answer lies.

25. $7.9 + 12.6 + 1.3$

26. $26.11 - 2.9$

27. $3.99 \times .96$

28. $49.2 \div 4.9$

29. $15.7 + .9 + 2.98$

30. $8.4 - .97$

31. 2.15×14.1

32. $4.92 \div 4.9$

33. $.09 + 15.6 + 23 + .007$

34. $50 - 15.912$

35. $.099 \times 10.1$

36. $5.01 \div 9.9$

THE BEST GUESS
(Fractions)

Which answer is best?

1. $\frac{1}{3} + 1\frac{9}{10}$

Less than	More than
2	2

2. $1\frac{1}{2} - \frac{7}{8}$

Less than	More than
1	1

3. $\frac{9}{10} \times \frac{7}{8}$

Less than	More than
1	1

4. $3\frac{3}{4} \div 2$

Less than	More than
2	2

5. $3\frac{1}{8} + 2\frac{1}{32}$

Less than	More than
$5\frac{1}{2}$	$5\frac{1}{2}$

6. $5\frac{2}{3} - 1\frac{1}{5}$

Less than	More than
4	4

7. $3\frac{1}{10} \times 4$

Less than	More than
12	12

8. $5\frac{9}{10} \div 1\frac{1}{8}$

Less than	More than
6	6

9. $2\frac{7}{8} + 3\frac{7}{8} + 4\frac{7}{8}$

Less than	More than
12	12

10. $8 - 2\frac{4}{5}$

Less than	More than
6	6

11. $3\frac{9}{10} \times 2\frac{9}{10}$

Less than	More than
7	7

12. $4\frac{1}{2} \div 5\frac{1}{2}$

Less than	More than
1	1

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The Best Guess (Fractions)

Mathematics teaching objectives:

- Estimate answers to problems involving fractions.
- Use rounding to help make estimates.

Problem-solving skills pupils might use:

- Make reasonable estimates.
- Recognize limits and eliminate possibilities.

Materials needed:

- Overhead transparencies for each lesson sheet

Comments and suggestions:

- These estimation problems could be used effectively as a beginning of the period activity. Since these exercises involve fractions, it may be best to use fewer problems in any one session than in the previous "best guess" sessions.
- Estimation should be an important part of the mathematics curriculum. Some authorities say that 75% of the adult non-occupational uses of math involve mental arithmetic and estimation. Frequently it is not necessary to have an exact answer, amount, or measurement. Pupils need many exposures to the estimation process. Some are reluctant to try. They would rather work a somewhat complicated problem in order to get "the" exact answer. The process needs to be taught (and practiced) using such individual skills as computing with single digits, rounding, and performing operations with powers and multiples of ten.
- This is a teacher-directed activity. Make overhead transparencies for each sheet. Project the problems on the screen but show only one at a time. Allow pupils a limited amount of time to respond--then give the answer, discuss if necessary and go on to another problem.
- You might want pupils to try to write down their answers before you give the correct answer. To motivate this, you could call it a "Beat the Teacher" activity. It is important, however, for pupils to use pencil and paper only to give their answers, not to do any computation whatsoever.

Answers:

1. More than 2
2. Less than 1
3. Less than 1
4. Less than 2
5. Less than 5½
6. More than 4
7. More than 12
8. Less than 6
9. Less than 12
10. Less than 6
11. More than 7
12. Less than 1
13. Between 1½ and 2
14. Between 2 and 2½
15. Between 15 and 15½
16. Between 3 and 3½
17. Between 2½ and 3
18. Between 4½ and 5
19. Between 3 and 4
20. Between 3½ and 4
21. Between 10 and 10½
22. Between 4 and 4½
23. Between ½ and 1
24. Between ½ and 1

Some possible responses:

25. A little less than 5
26. A little more than 0
27. About 12
28. A little more than 0
29. Between 2 and 3
30. A little less than 2
31. Between 8 and 9
32. About 8
33. Between 7 and 8
34. Between 3 and 4
35. About 14
36. About 5

13. $\frac{1}{2} + 1\frac{1}{3}$

Between 1 and $1\frac{1}{2}$

$1\frac{1}{2}$ and 2

2 and $2\frac{1}{2}$

14. $3\frac{1}{6} - 1\frac{1}{24}$

Between $1\frac{1}{2}$ and 2

2 and $2\frac{1}{2}$

$2\frac{1}{2}$ and 3

15. $3 \times 5\frac{1}{32}$

Between 15 and $15\frac{1}{2}$

$15\frac{1}{2}$ and 16

16 and $16\frac{1}{2}$

16. $9\frac{1}{8} \div 3$

Between $2\frac{1}{2}$ and 3

3 and $3\frac{1}{2}$

$3\frac{1}{2}$ and 4

17. $\frac{7}{8} + \frac{15}{16} + \frac{31}{32}$

Between $1\frac{1}{2}$ and 2

2 and $2\frac{1}{2}$

$2\frac{1}{2}$ and 3

18. $7\frac{3}{10} - 2\frac{1}{2}$

Between $4\frac{1}{2}$ and 5

5 and $5\frac{1}{2}$

$5\frac{1}{2}$ and 6

19. $1\frac{9}{10} \times 1\frac{9}{10}$

Between 1 and 2
2 and 3
3 and 4

20. $7\frac{19}{20} \div 2\frac{1}{10}$

Between $3\frac{1}{2}$ and 4
4 and $4\frac{1}{2}$
$4\frac{1}{2}$ and 5

21. $7\frac{1}{2} + 2\frac{15}{16}$

Between 9 and $9\frac{1}{2}$
$9\frac{1}{2}$ and 10
10 and $10\frac{1}{2}$

22. $10 - 5\frac{7}{8}$

Between 4 and $4\frac{1}{2}$
$4\frac{1}{2}$ and 5
5 and $5\frac{1}{2}$

23. $\frac{1}{2} \times 1\frac{9}{10}$

Between $\frac{1}{2}$ and 1
1 and $1\frac{1}{2}$
$1\frac{1}{2}$ and 2

24. $2\frac{1}{2} \div 2\frac{3}{4}$

Between $\frac{1}{2}$ and 1
1 and $1\frac{1}{2}$
$1\frac{1}{2}$ and 2

The Best Guess (Fractions) (cont.)

Estimate the answer to each exercise. It may help to first determine limits within which the answer lies.

25. $1\frac{1}{2} + 3\frac{3}{10}$

26. $3\frac{1}{2} - 3\frac{1}{8}$

27. $4 \times 2\frac{7}{8}$

28. $4\frac{1}{2} \div 4\frac{1}{4}$

29. $\frac{1}{2} + \frac{7}{8} + \frac{7}{8}$

30. $4\frac{1}{8} - 2\frac{2}{8}$

31. $2\frac{9}{10} \times 2\frac{9}{10}$

32. $16 \div 2\frac{1}{10}$

33. $5\frac{7}{8} + 1\frac{7}{8}$

34. $12 - 8\frac{2}{5}$

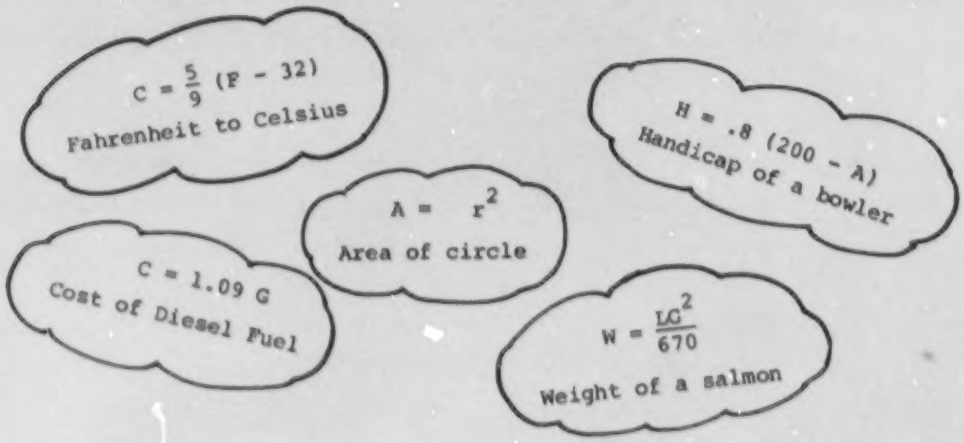
35. $1\frac{7}{8} \times 7\frac{1}{10}$

36. $9\frac{9}{10} \div 2\frac{1}{5}$

Grade 8

III. VARIATION

III. VARIATION



Formulas, tables, and graphs often have a close relationship to the "real world" and frequently can be used in making "real world" predictions. These topics can be used to show how one thing (variable) changes (varies) as another thing (variable) changes. The situations for studying change in this section involve two variables and are confined to those likely to be within the "world of experience" of eighth graders.

Another name for this unit could have been "An Introduction to Elementary Functions: An Intuitive Approach." However, the writers hesitated to use the term, function, at this early stage. The word is confusing since its use in a context familiar to pupils has a different meaning.

The lessons informally introduce pupils to such terminology as variable, vary in the same sense and vary in the opposite sense, origin, and axes.

The pupil pages provide an introduction to

- evaluating formulas.
- making tables from a formula
- writing a formula for a table of values
- creating a story problem for a formula
- matching a graph with its formula
- describing the type of change suggested by a table, graph, or formula
- predicting whether or not the graph for a table suggests a straight or curved line.

After studying the unit, pupils should realize

. formulas even though useful seldom, if ever, exactly describe data collected from the real world.

. caution should be exercised in making "real world" predictions from tables, graphs, and formulas.

. formulas have a range of applicability.

This study of variation will be more meaningful if pupils have an opportunity to collect "real world" data. Some actual experiments that could be done are

. how the rebound of a tennis ball changes as the dropping height changes.

. how the period of a pendulum changes as the length of the string changes.

This unit on change and variation could be presented early in the school year and be the beginning of a strand that holds together much of the eighth-grade program. Geometric formulas, graphing, Pythagorean theorems, statistics, and equation solving all are somewhat related to variation and could be follow-up units during the year.

The lessons in the unit should be done in the order given. Also, ample class time should be allowed so pupils might receive assistance as difficulties arise.

HAMBURGERS FOR SLIM

Slim used to eat lots of hamburgers. Sometimes he had only enough money for a few; sometimes he had enough money for a weekly supply.

1. a. Fill in the table to show how the cost changes as Slim buys more hamburgers.

Hamburgers 29¢ each

A formula is a statement of how things vary or change.

This formula will help you.

Cost = 39 x Number of Hamburgers

Number of Hamburgers	1	2	5	8	10				
Cost in Cents						117	234	429	819

b. Fill in the blanks. Use the words <u>increases</u> and/or <u>decreases</u>.

The table shows the cost _____ as the number of hamburgers _____.

Explain how the formula shows this type of variation.

2. Hamburgers 59¢ each

Cost = 59 x H

A table is a chart showing how things vary or change.

a. Complete the table.

H	1	2	3	4	5		9		
C						590		649	472

b. Compare the table and formula here with those in problem 1. How are they alike? How different?

Hamburgers For Slim

Mathematics teaching objectives:

- Use formulas to show change.
- Use tables to show change.
- Interpret graphs which show change.

Problem-solving skills pupils might use

- Use a systematic list.
- Look for patterns.
- Work backwards.
- Create new problems by varying ones already solved.

Materials needed:

- None

Comments and suggestions:

- Do this lesson in two parts--problems 1-4 on one day; 5-7 on another day.
- Review rules for order of operations.
- Have pupils discuss the various ways they completed the tables. For example, in table 2, the cost of 9 hamburgers can be found by adding the costs of 4 hamburgers and 5 hamburgers.
- Point out the similarities between the formulas and tables for problems 2 and 4.
- This is the first of several lessons on variation and change. Pupils should recognize that in the situations presented here, an increase in one amount (the number of hamburgers) will result in an increase in the other amount (the cost). Ask pupils to give examples of other situations where this kind of variation will occur.

Answers:

1. a.

Number	1	2	5	8	10	3	6	11	21
Cost	39	78	195	312	390	117	234	429	819

b. increases, increases or decreases, decreases. Sometimes this change is described as "varying in the same sense.")

The formula for cost is the product of 39 and another number, depending upon how many hamburgers purchased. This product is always greater when this number is greater.

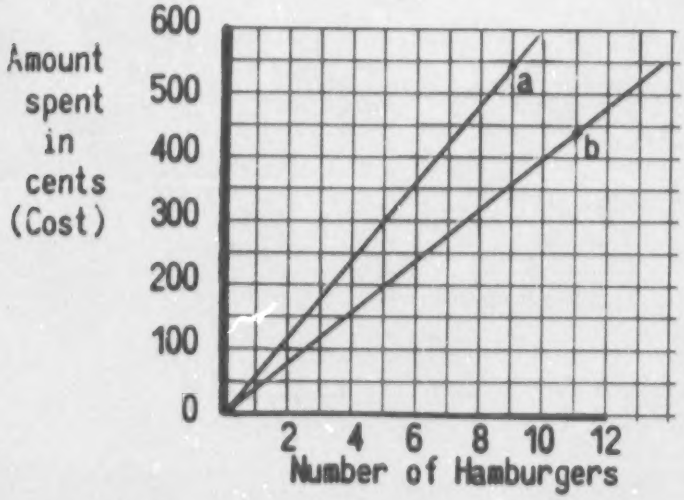
2. a.

H	1	2	3	4	5	10	9	11	8
C	59	118	177	236	295	590	531	649	472

b. Comparison of tables (1-a and 2a):
H is used instead of Number of Hamburgers.
C is used instead of Cost.
Numbers for Number of Hamburgers in each table are not ordered from smallest to largest. Same is true for C.

Comparison of formulas (1-b and 2b):
The factor that does not change is larger.
The formula for cost in each case is a product.

3.



The lines a and b are "pictures" or graphs of the formulas in problems 1 and 2. Which graph goes with each problem? How can you tell?

4. Examine the table. How much will each hamburger cost Slim? Write a formula and complete the table.

H	0	1	2	3	4	5	6	10		15
C							330		605	

5. Slim has decided that he needs one 30¢ cola to drink as he eats his 59¢ hamburgers.

C = 59H + 30 Note: 59H means 59 x H.

a. Complete this table.

H	1	2	3	4	5	6	7			15
C								679	738	

Answers (cont.)

3. Line a is for formula in problem 2 - Cost = 59 x H

   Line b is for formula in problem 1 - Cost = 39 x H

   Line a, the steeper line, goes with the more expensive hamburger.

4. Each hamburger will cost 55¢. (330 ÷ 6) C = 55 x H

H	0	1	2	3	4	5	6	10	11	15
C	0	55	110	165	220	275	330	550	605	825

5. a.

H	1	2	3	4	5	6	7	11	12	15
C	89	148	207	266	325	384	443	679	738	915

   b. Graph a and Graph b are both straight lines; also both lines are "equally steep," therefore parallel lines.

   Graph a intersects both "heavy black lines" at their point of intersection.

   The lines are parallel because in all cases only one 30¢ coke was purchased.

   Note to teacher: You may wish to introduce the terms origin, amount spent axis, and number of hamburgers axis.

6. a. A comparison of the formula C = 89H + 35 with C = 59H + 30 shows that the price of a hamburger increased 30¢ and the cost of a coke increased 5¢.

   Note to teachers: You may wish to introduce the terms variable and constant. In the formula C = 89H + 35, C and H can have different number values depending on the circumstances. Therefore, they are called variables. In this formula, the numbers 89 and 35 are constants.

   b. 22 hamburgers. (The problem can be solved by working backwards--first subtract 35¢, then divide by 89. He still has 7¢ left.)

7. Share some of the problems pupils have created with the rest of the class. Here are two problems that fit the formulas:

   a. Slim bought some hot dogs for 49¢ each. He also bought one 39¢ cola and one 55¢ shake.

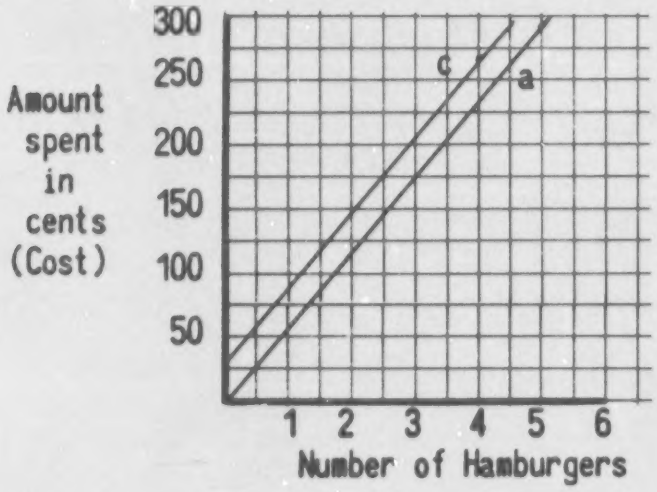
   b. Slim bought some hamburgers for 59¢ each and some colas for 35¢ each.

Extension:

H	8*	6	4	2	0*
F	0	3	6	9	12

*Pupils may not consider these solutions.

5. b. Line a is a graph of the hamburger formula in problem 2. Line c is the graph of the hamburger and cola formula in problem 5a.



Compare graph c and graph a. How are they alike? How are they different? Explain why line c is parallel to line a.

6. Prices have increased. Here is Slim's new formula:

$C = 89H + 35$ Note: 89H means 89 x H.

a. How much were the price increases? See problem 5.

b. Slim has $20 to spend. How many hamburgers can he buy? Remember, he also buys one cola.

7. Create your own problem for each of these formulas:

   a. $C = 49H + 39 + 55$

   b. $C = 59H + 35P$

EXTENSION

The "Good Ol' Days"

Slim remembers the "good ol' days." Hamburgers were 15¢ and fries were a dime.

Show how Slim could have spent exactly $1.20 on hamburgers and fries.

List as many solutions as you can. (Find more than 3 solutions.)

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FORMULAS

Sammy was pleased with himself. He had just figured out a new formula. Sammy named the formula

Ideal Weight For Everyone

$W = 5H - 190$ H is the person's height in inches.
W is the weight in pounds.

Is this a reasonable formula to use? Let's first see how it works for Sammy. Then we'll see how it works out for you.

Class Exercises

1. Sammy is 64 inches tall and weighs 128 pounds. According to the formula, how much should Sammy weigh?

2. What is your own height in inches? According to the formula, how much should you weigh?

3. Mr. Garcia is 6 feet tall and weighs 173 pounds. According to the formula, how much should he weigh?

4. Karen's little sister is 40" tall. According to the formula, how much should she weigh?

5. Should Sammy change the name of the formula? Why?

Exercises

1. Here is a Height-Weight formula which Karen made up:

$W = 4H - 150$

A formula is a statement of how things vary.

A table is a chart showing how things vary.

Use Karen's formula to complete the table of values.

H	56	60	64	68
W				

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Formulas

Mathematics teaching objectives:

- Use formulas and tables to show variation.
- Interpret line graphs.

Problem-solving skills pupils might use:

- Recognize the limits of a formula.
- Use a systematic list.
- Look for patterns.

Materials needed:

- None

Comments and suggestions:

- In this lesson the numbers for one variable are ordered from the smallest to largest. Discuss the advantage of such systematic listing.
- Review rules for order of operations and use of parentheses.
- The "height-weight" formulas and "hours of sleep" formula only seem to work reasonably well within certain limitations. This point needs to be emphasized. Also, the idea of what is meant by "reasonably well" should be discussed.
- The "height-weight" formulas behave the same as the "hamburger" formulas in the last lesson--that is, as one amount increases, the other amount also increases. The "hours of sleep" formula shows a different kind of variation. In this situation as one amount increases, the other decreases.
- Use terms such as variable, constant, origin, axis of a graph. The emphasis should be casual rather than formal.

Answers:

Class Exercises

1. 130 lbs.
2. Answers will vary.
3. 170 lbs.
4. 10 lbs.
5. Probably. At least the word "everyone" should be changed. The formula was reasonably accurate for Sammy and Mr. Garcia, but not so for Karen's sister.

Exercises

1.

H	56	60	64	68
W	74	90	106	122

Formulas (cont.)

2. Use Karen's formula.

   a. According to Karen's formula, how much should you weigh?

   b. According to her formula, how much should Karen's little sister weigh? Remember, Karen's little sister is 40" tall.

   c. Is Karen's formula better than Sammy's? Why?

   d. Karen called her formula the "Ideal Weight for Junior High Pupils." Do you think this is a reasonable name? Explain.

3. Below are some height-weight formulas made up by Sammy's friends.

   Ross $W = 2(H - 8)$ Jill $W = 180 - H$ Sue $W = \frac{H}{2} + 90$

   a. Which formula seems to work best for you? Explain.

   b. Use each formula to complete the tables below.

Ross

H	56	60	64	68
W				

H	56	60	64	68
W				

H	56	60	64	68
W				

Fill in the blanks. Use the words increases and/or decreases.

   c. Ross' table shows that the height _____ as the weight _____. Explain how the formula shows this.

   d. Jill's table shows that the height _____ as the weight _____. Explain how the formula shows this type of variation.

   e. Sue's table shows that the height _____ as the weight _____. Explain how the formula shows this.

   f. Which formula seems to be the worst one to use? Why?

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Answers (cont.)

2. a. Answers will vary.

   b. 10 lbs.

   c. Answers will vary. It does seem to be a better formula for certain age groups, but not for Karen's.

   d. Perhaps. It does seem to fit junior high pupils better than other age groups.

3. a. Answers will vary.

   b.

Ross				
H	56	60	64	68
W	96	104	112	120

Jill				
H	56	60	64	68
W	124	120	116	112

Sue				
H	56	60	64	68
W	118	120	122	124

   c. increases, increases or decreases, decreases. (H - 8) increases as H decreases. Therefore, the product of 2 and (H - 8) also increases. Another example of variation in the same sense.

   d. increases, decreases or decreases, increases. If you subtract a larger number from a given number, the answer is smaller. An example of variation in the opposite sense.

   e. increases, increases or decreases, decreases. $\frac{H}{2}$ increases as H increases. Therefore, the sum of $\frac{H}{2}$ and 90 also increases. Another example of variation in the same sense.

   f. Jill's. As you get taller, according to the formula you will lose weight.

4. a. line c
   b. line a
   c. line b
   d. The graph "goes down" rather than "up" like the other two.

   Note to the teacher: The "zig zag" lines on the axes indicate that part of the graph is missing. Often, not showing the zero points (origin) on both axes is misleading. Graphs in newspapers and magazines (especially advertisements) frequently fail to show their origin. Post such graphs on the bulletin board and discuss whether or not they are misleading.

5. a.

A	2	6	10	14	18
H	16	14	12	10	8

   b. 0 hours
   c. Answers will vary.
   d. The formula works reasonably well from ages 2 to 20.

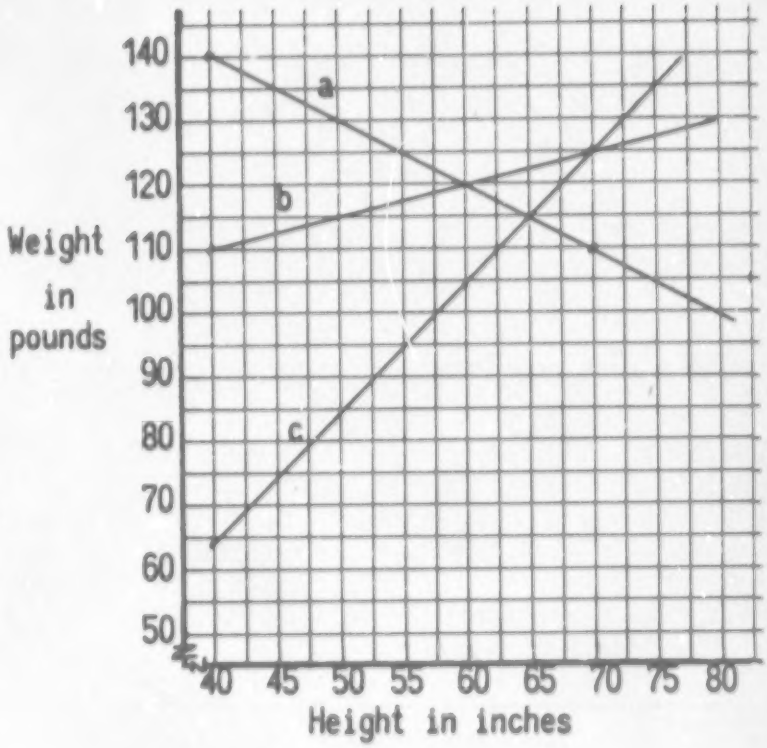
4. The lines a, b, and c are graphs for the three formulas in problem 3. Which graph is for:

   a. Ross' formula?

   b. Jill's formula?

   c. Sue's formula?

   d. How can you tell that line a is not a graph for showing how the weight of a person changes with his/her height?



5. Coach Young told her basketball team that they should be getting more sleep. She used an "Hours of Sleep Formula" to emphasize her point.

   $H = (34 - A) \div 2$

   A is the person's age in years.
   H is the hours of sleep per day.

   a. Use the formula to complete the table of values.

A	2	6	10	14	18
H					

   b. Mr. Garcia is 34 years old. According to the formula, how much sleep should he be getting?

   c. According to the formula, how much sleep should your mother or father be getting?

   d. Within what limits does the formula seem to work reasonably well?

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BELIEVE IT OR NOT FORMULAS

Would you believe that a cricket can be used as a thermometer? The warmer it is, the faster a cricket chirps. This formula gives a close approximation:

$T = \frac{1}{4}C + 37$ C is the number of chirps in one minute.
T is the temperature in Fahrenheit.

Formulas are statements of variation or change.
The weight-height formula suggests that
- a person's weight increases as he/she gets taller.

The sleep-age formula suggests that
- the number of hours of sleep a person needs decreases as he/she gets older.

According to the cricket chirps formula, how does the temperature vary as the cricket chirps faster? Let's find out.

1. Use the cricket formula to complete this table.

C	56	64	72	84	120	160
T						

   a. The temperature is 54^0. Estimate the number of chirps a cricket would make. Check your estimate by using the formula.

   b. Estimate the number of chirps if the temperature is 72^0. Check your estimate by using the formula.

2. A certain kind of ant also can be used as a thermometer. The warmer it is, the faster the ant runs.

   $T = 11R + 39$ R is the speed in inches per minute.
   T is the temperature in Fahrenheit.

   Use the formula to complete this table.

R	2	3	4	5	6
T					

   a. If the temperature is 66^0, <u>about</u> how fast does the ant run?

   b. If a cricket chirps 160 times a minute, <u>about</u> how fast does the ant run?

Believe It Or Not Formulas

Mathematics teaching objectives:

- Use formulas and tables to show change.

Problem-solving skills pupils might use:

- Use a systematic list.
- Look for patterns.
- Make reasonable estimates.
- Search for needed information.

Materials needed:

- None

Comments and suggestions:

- Use terms such as variable and constant. The emphasis should be casual rather than formal.
- Remind pupils that reasonable estimates are all that are expected in several of the problems (see 2-a and 2-b).
- Review order of operations, especially as they apply to expressions like $16T^2$.
- You may wish to perform an experiment like the one suggested in problem 4. Be certain to discuss limitations due to air resistance (e.g., a feather and a book).
- The situation in problem 4 shows a different kind of variation or change. Although the variable, D, increases as T increases, the amount of increase for D is NOT uniform for every uniform increase for T. Use the table in problem 4 to show this.
- The comparable metric formulas for problems 1 and 2 are:

1. $T = \frac{C}{7} + 3$ T is the temperature in CELSIUS.

2. $T = \frac{5R + 8}{2}$ R is the rate in cm/min.
T is the temperature in CELSIUS.

Answers:

1.

C	56	64	72	84	120	160
T	51	53	55	58	67	77

a. 68 chirps

b. 140 chirps

2.

R	2	3	4	5	6
T	61	72	83	94	105

a. About $2\frac{1}{2}$ in./min.

b. A little less than $3\frac{1}{2}$ in./min.

3. It's hard to believe that in one second a space capsule can travel approximately 5 miles. The following formula will help us see how the distance varies as the time increases.

   $D = 5T$ T is the time in seconds.
   D is the distance in miles.

   Complete this table.

T	1	2	3	5	10	30	60
D							

   a. How far will the capsule travel in one minute?

   b. How far in one hour?

   c. Describe how the distance traveled by the space capsule changes as the time for travel changes. How could you have predicted this by studying the formula?

   d. How long would it take the space capsule to go from over New York to over San Francisco?

4. Suppose you dropped a pencil and a penny from the same height at the same time. Would you believe they will reach the floor at the same time? Try it.

   The following formula can be used to find the distance an object falls (no matter how heavy the object is).

   $D = 16T^2$ T is the number of seconds.
   D is the distance in feet.

   This means
   16 times T times T.

   Complete this table.

T	1	2	3	4	5
D					

   a. About how many seconds should it take a boulder to drop 300 feet?

   b. The Empire State Building is 1250 feet. About how long should it take a penny to reach the ground below?

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Answers: (cont.)

3.

T	1	2	3	5	10	30	60
D	5	10	15	25	50	150	300

a. 300 miles

b. 18,000 miles

c. As the time T increases, the distance D also increases. The product of 5 and T is larger for larger values of T.

d. 514 to 515 seconds (about $8\frac{1}{2}$ minutes).

Note to teacher: According to the WORLD ALMANAC the air distance from New York to San Franciso is 2,572 miles. Have pupils look up this information for themselves.

4.

T	1	2	3	4	5
D	16	64	144	256	400

a. Between 4.3 and 4.4 seconds

b. A little less than 9 seconds

CARS AND BIKES

Mrs. Schafer is pleased with her new car. It is easy to see why. On a recent business trip of 248 miles she discovered that she had used only 8 gallons of gas.

Mrs. Schafer is looking forward to her next trip of 345 miles. She wonders if she'll be able to complete the entire trip without refueling.

Before Mrs. Schafer can solve this problem, what additional information must she know?

Class Exercises

1. Mrs. Schafer's tank holds 11 gallons. Perhaps if we complete this table it will help us solve the problem.

Gallons	1	2	4	8
Miles				248

2. Do you think Mrs. Schafer will be able to drive 345 miles on 11 gallons? Explain.

3. Write a formula for this problem. Use G for the number of gallons and M for the number of miles.

Exercises

1. Mrs. Schafer usually buys gasoline at $1.29 per gallon.
   a. At $1.29 what will be the cost of 5 gallons? 10 gallons?
   b. Write a formula for this situation. Use G for the number of gallons and C for cost in dollars.
   c. About how much will it cost Mrs. Schafer to make a trip of 345 miles?

Cars And Bikes

Mathematics teaching objectives:

- Write formulas that fit tables of value.
- Use formulas and tables to show change.
- Interpret graphs which show change.

Problem-solving skills pupils might use:

- Use a systematic list.
- Look for patterns.
- Work backwards.

Materials needed:

- None

Comments and suggestions:

- To determine if Mrs. Schafer can travel 345 miles, she needs to know how many gallons her tank holds.
- The opening problem may provide opportunity for further discussion regarding fuel shortage and costs.
- Problem 2 is another example of variation in the opposite sense--as one variable increases, the other decreases. In this situation, as one quantity doubles, the other is halved (an inverse variation).

Answers:

Class Exercises

1.

G	1	2	4	8
M	31	62	124	248

2. Probably not, at least at the rate of 31 miles/gallon

3. M = 31G

Exercises

1. a. $6.45; $12.90
   b. C = 1.29 G
   c. A little more than $14.00

2. The distance between Elmstown and Mill City is 48 miles.

   a. Suppose Becki bicycles at the rate of 12 mph. How long will it take her to cycle from one town to the other?

   b. Suppose Martha bicycles at the rate of 16 mph. How long will it take her to make the trip?

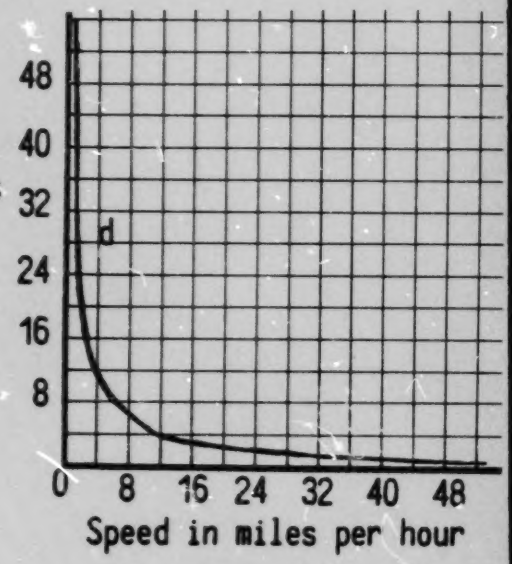
   c. Adam drives at the rate of 48 mph. How long will it take him to drive from Elmstown to Mill City?

   d. These results can be organized in a table. Complete the table.

Speed	48	24	16	12	8	6	4	3	2	1
Time	1		3	4						

   e. Write a formula for this situation. Use S for speed in miles per hour and T for time in hours.

   f. The line d is a graph of the speed, time, and distance formula. Mark the points on the graph that go with the speeds and times shown in the table. Suppose the distance were 96 rather than 48. Sketch in.



   g. Fill in the blanks. Use the words <u>increases</u> and/or <u>decreases</u>.

   The graph shows that the speed in miles per hour to travel 48 miles _____ as the time traveled _____.

   h. The table for Coach Young's sleep formula is

A	2	6	10	14	18
H	16	14	12	10	8

A is a person's age in years.
H is the hours of sleep per day.

Compare the table for the distance formula (problem 2a) with the table above. How are they alike? How different? Will the graph of the sleep formula be a curved line or a straight line? Explain.

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Answers:

Exercises (cont.)

2. a. 4 hours

b. 3 hours

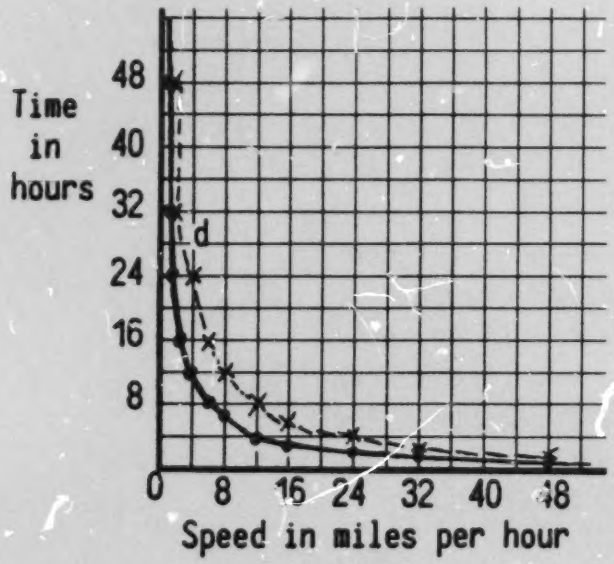
c. 1 hour

d.

S	48	24	16	12	8	6	4	3	2	1
T	1	2	3	4	6	8	12	16	24	48

e. $ST = 48$ or $T = \frac{48}{S}$ or $S = \frac{48}{T}$

f.



g. decreases, increases or increases, decreases.

Note: An example of variation in the opposite sense. As the speed is cut in half, the time is doubled.

h. Both are examples of variation in the opposite sense. As the variable (time) increases a uniform amount, the decrease in the variation (distance) IS NOT a uniform amount. The graph for the sleep formula is a straight line. As the variable (age) increases a uniform amount, the decrease in the variable (hours) IS a uniform amount.

3. a. Answers will vary.
b. 216 feet
c. 384 feet
d. The car may have been speeding or close to it. According to the formula, a speed of 55 mph will result in a stopping distance of 181.5 feet.
e. A curved line. Uniform increases in the variable R produces a non-uniform increases in the distance. Suggestions: make a table with the class to show this.

3. A car is speeding down the road at 60 mph. The driver sees an accident and slams on the brakes. After he applies the brakes, how far do you think he'll travel before stopping? (Assume the car, the road, and the driver are in good condition.)

   a. Make a guess: 50 feet; 100 feet; 150 feet; 200 feet

   b. This formula gives a reasonable approximation.

   $D = .06R^2$

   R is the speed in mph.
   D is the distance in feet that the car travels.

   Use the formula to determine the "stopping distance" when the speed is 60 mph. How close was your guess?

   c. Suppose the speed is 80 mph. Now what is the stopping distance?

   d. A policeman measures the skid distance of a car involved in an accident. She finds it to be 190 feet. Do you think the car was speeding? Explain.

   e. If a graph were made for $D = .06R^2$, would it be a curved line or a straight line? Explain.

Grade 8

IV. INTEGER SENSE

TEACHER COMMENTARY - GRADE 8

IV. INTEGER SENSE

Negative numbers are not new ideas for eighth graders. They have had experiences with temperatures "below zero" and with "going in the hole" when playing certain games. Pupils should be able to work the lessons in this section based upon these experiences and their problem-solving skills. The overall objective is for pupils to gain intuitive understandings, not to develop rules for operations with positive and negative numbers.

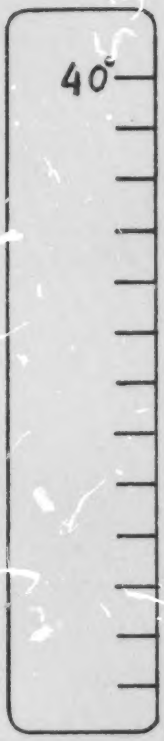
We would recommend that a formal approach using the usual rules found in an algebra text be deferred until the algebra year. When rules go counter to the pupil's intuition, care needs to be exercised in using these rules at the initial stages of instruction. What pupils learn should at all times make sense to them. Expressions such as "in order to add these two numbers, you subtract" or "in order to subtract, you add" or "the product of two negative numbers is a positive number" does not initially make sense to the pupils. If your course of study calls for a formal treatment of the usual rules, you will need to supplement the treatment here with more lessons. This section was designed to take about a week; a unit involving the usual rules probably would take three weeks. We recommend that the lessons be done in the indicated order.

THERMOMETER READINGS

Numbers are missing from the Celsius thermometer pictured at the right.

The marks are in 5^0 intervals. See if you can properly number the remaining marks.

Use the thermometer to help you work the following exercises.



1. Arrange these temperature readings from coldest to warmest:

   $^-10^0$, 15^0, $^-3^0$, 0^0, 21^0, $^-2^0$

2. Suppose the present temperature is 5^0.
   What is the reading if the temperature
   a. rises 20^0? b. rises 5^0? c. falls 17^0?

3. Suppose the present temperature is $^-5^0$ (5 degrees below zero).
   What is the reading if the temperature
   a. rises 30^0? b. rises 5^0? c. falls 14^0?

4. What temperature reading would be halfway between
   a. 0^0 and 40^0? c. $^-15^0$ and 5^0? e. 2^0 and $^-10^0$?
   b. $^-10^0$ and 10^0? d. 15^0 and $^-5^0$? f. 5^0 and $^-13^0$?

5. Which temperature is farther from $^-2^0$?
   a. 7^0 or $^-10^0$ b. 2^0 or $^-8^0$ c. 1^0 or $^-5^0$

Thermometer Readings

Mathematics teaching objectives:

- Solve problems using integers in situations involving the thermometer.
- Build intuitive background involving operations with integers.

Problem-solving skills pupils might use:

- Make and use a drawing.
- Follow written directions.
- Guess and check.

Materials needed:

- None

Comments and suggestions:

- We recommend that a formal approach using the usual rules for operations with integers not be given in this pre-algebra year.
- Do the lesson during class so that you can give assistance when needed.
- After a short introduction involving the making of a thermometer pupils should be able to proceed with the lesson independently. This visual prop will help them obtain the answers without a formal knowledge of operations with integers.
- During the culminating discussion, remind the class that "changes" can be found by subtracting one reading from another. This is true with integers as well as with whole numbers.
- The extensions provide pupils practice with both integers and logic. The use of appropriately marked slips of papers as markers for the requested points on the thermometer will aid pupils in solving these extensions. Allow opportunities for pupils to share their "created puzzles" with classmates.

Answers:

1. -10°, -3°, [illegible] 0°, 15°, 21°
2. a. 25° b. 10° c. -12°
3. a. 25° b. 0° c. -19°

6. Study the first two lines of the table. Then complete the rest of the table.

8:00 a.m.	8:00 p.m.	Change in Temperature from 3:00 a.m. to 8:00 p.m.
5^0	-5^0	-10^0
-5^0	5^0	10^0
3^0	-8^0	?
-8^0	3^0	?
-2^0	-13^0	?
-13^0	-2^0	?
-1^0	?	15^0
?	-2^0	-8^0
5^0	?	12^0
?	-5^0	4^0

7. At 7:00 p.m. the temperature in Coldspot was -2^0. The hourly changes from 7:00 p.m. to midnight were

   3^0, 4^0, -2^0, -3^0, -4^0.

   What was the temperature at midnight?

8. The temperature for the past 7 hours has been changing at the rate of -2^0 each hour. The weatherman predicts it will continue changing like this for the next 7 hours. The present reading is 0^0.

   a. What was the reading 7 hours ago?
   b. What temperature is predicted for 7 hours from now?
   c. When was the reading 6^0?
   d. When is the temperature expected to be -8^0?

Answers: (cont.)

4. a. 20° b. 0° c. -5° d. 5° e. -4° f. -4°

5. a. 7° b. -8° c. Neither

6.

8:00 a.m.	8:00 p.m.	Change in Temperature from 8:00 a.m. to 8:00 p.m.
5°	-5°	-10°
-5°	5°	10°
3°	-8°	(-11°)
-8°	3°	(11°)
-2°	-13°	(-11°)
-13°	-2°	(11°)
-1°	(14°)	15°
(6°)	-2°	-8°
5°	(17°)	12°
(-9°)	-5°	4°

7. -4°

8. a. 14° b. -14° c. 3 hours ago d. 4 hours from now.

Extensions:

1. The points from top to bottom are A, E, D, B, C
2. N
3. Answers will vary.

EXTENSIONS

1. A, B, C, D, and E correspond to the five points marked on the thermometer at the right. Use these clues to locate the points. (Hint: zero is not one of the dots given on the line.)

   B and D are negative.
   D is warmer than C.
   B is warmer than C.
   E and D are the same distance from zero.
   E is colder than A.

2. M, N, P, and Q represent four different temperature readings. Draw a thermometer and locate the points to fit the clues.

   Clues: Q is the coldest.
   M and Q are the same distance from zero.
   P is positive.
   P is closer to Q than to N.

3. Make up and solve your own "temperature" puzzle. Don't give too many clues; don't give too few.

NEGATIVE NUMBERS

1. In the 15th Century, European flour merchants used positive and negative numbers. On a flour barrel $^{+}5$ meant that it was 5 pounds overweight; $^{-}5$ meant that it was 5 pounds underweight.

   a. The following numbers were on 100-pound flour barrels.

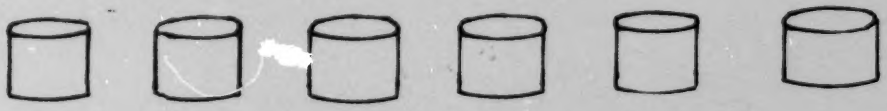
   Did the five barrels contain more or less than 500 pounds? How much more or less?

   b. These numbers were on eight 100-pound barrels:

   $^{-}3,\ ^{-}5,\ ^{+}9,\ ^{+}1,\ ^{-}8,\ ^{+}3,\ ^{+}5,\ ^{+}2$

   Did the barrels contain more or less than 800 pounds? How much more or less?

   c. The drawings below show six 100-pound barrels. The total weight in the barrels is 13 pounds less than 600 pounds. Suppose that each barrel is marked with a different number. Show a way that the barrels could be marked. (Keep your numbers between $^{-}10$ and $^{+}10$.)



2. Tank is a football fullback. Sometimes he gains yardage ($^{+}5$ means a five-yard gain). Sometimes he loses yardage ($^{-}8$ means an eight-yard loss).

   a. Determine Tank's total yardage in each of these games.

   Game 1: $^{+}3,\ ^{+}5,\ ^{+}6,\ ^{+}1,\ ^{+}13$
   Game 2: $^{-}2,\ ^{-}6,\ ^{-}1,\ ^{-}2,\ ^{-}4$
   Game 3: $^{-}4,\ ^{+}4,\ ^{-}3,\ ^{+}3,\ ^{+}10,\ ^{-}13$
   Game 4: $^{-}5,\ ^{+}3,\ ^{+}6,\ ^{-}2,\ ^{+}8,\ ^{-}6,\ ^{+}9,\ ^{-}1$
   Game 5: (See if you can figure out an easy way to do this one.)
   $^{+}6,\ ^{-}7,\ ^{-}5,\ ^{+}10,\ ^{-}3,\ ^{-}8,\ ^{+}5,\ ^{-}10,\ ^{+}8,\ ^{+}3,\ ^{+}7,\ ^{-}11,\ ^{-}6$

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Negative Numbers

Mathematics teaching objectives:

- Solve problems using integers in situations involving barrels of flour, football, and games.
- Extend the concept of "total" to include negative addends.
- Build intuitive background involving addition of integers.

Problem-solving skills pupils might use:

- Follow written directions.
- Make a systematic list.

Materials needed:

- None

Comments and suggestions:

- Give out the lesson with the expectation that pupils will get their directions from reading.
- Later in the lesson you might choose to check how well the pupils are doing and to remind them of the importance of using the skill "make a systematic list" in solving problems 3 and 4.
- Make pupils aware of the new interpretation placed upon "total" and the operation "addition." The total when applied to integers can be a number less than one (or more) of the addends, e.g., $9 + (^{-}8) + 3 = 4$. Notice the total, 4, is less than the addend, 9.)

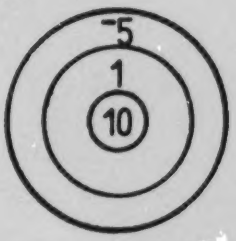
Answers:

1. a. 2 pounds less b. 4 pounds more c. Answers will vary.
2. a. Game 1: $^{+}28$
   Game 2: $^{-}15$
   *Game 3: -3
   Game 4: $^{+}12$
   Game 5: $^{-}11$
   b. Answers will vary.
   c. Any number greater than $^{+}9$.
   *Notice in this game the total is less than three of the addends. This does not happen in whole number arithmetic.

b. In Game 6, Tank's total yardage was $^{-}16$ yards for 8 carries. Show how this is possible if the yardage for each carry was different.

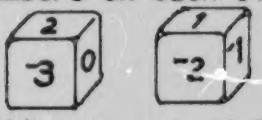
c. In the final game, Tank's first seven carries were $^{-}3$, $^{-}6$, $^{+}4$, $^{-}3$, $^{-}7$, $^{+}8$, $^{-}2$. How much would he have to gain on his eighth carry to have a positive total?

3. Teri made up a game of darts. Everyone was given a chance to throw 3 darts. If all 3 darts hit the target, list all possible scores that could result.

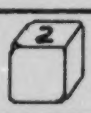
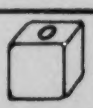


4. Greg made two 6-sided dice with these numbers on each of them:

$^{-}3$, $^{-}2$, $^{-}1$, 0, 1, 2



Suppose the pair of dice is tossed.
The diagram below shows three ways of getting a sum of 2.

1st die	2nd die	
		0 + 2
		2 + 0
		1 + 1

a. List all the ways to get a sum of $^{-}4$.

b. List all the ways to get a sum of 0.

c. Suppose the pair of dice was tossed 100 times. Which sum would probably be the most popular? Explain.

EXTENSION

Suppose Greg made another die like the ones he already has. If he rolled all three, list all the ways he might get a total of

a. $^{-}9$ b. $^{-}8$ c. $^{-}7$.

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Answers: (cont.)

3.

10	1	$^{-}5$	Score
✓✓✓			30
✓✓	✓		21
✓✓		✓	15
✓	✓✓		12
✓	✓	✓	6
✓		✓✓	0
	✓✓✓		3
	✓✓	✓	-3
	✓	✓✓	-9
		✓✓✓	-15

4. a. $^{-}1 + {}^{-}3$, $^{-}2 + {}^{-}2$, $^{-}3 + {}^{-}1$

   b. $2 + {}^{-}2$, $1 + {}^{-}1$, $0 + 0$, $^{-}1 + 1$, $^{-}2 + 2$

   c. A sum of $^{-}1$ is most likely.
   $2 + {}^{-}3$, $1 + {}^{-}2$, $0 + {}^{-}1$, $^{-}1 + 0$, $^{-}2 + 1$, $^{-}3 + 2$

Extension:

a. $^{-}3 + {}^{-}3 + {}^{-}3$

b. $^{-}3 + {}^{-}3 + {}^{-}2$, $^{-}3 + {}^{-}2 + {}^{-}3$, $^{-}2 + {}^{-}3 + {}^{-}3$

$^{-}3 + {}^{-}3 + {}^{-}1$ $^{-}2 + {}^{-}2 + {}^{-}3$

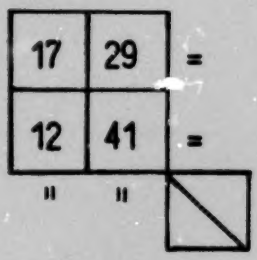
$^{-}3 + {}^{-}1 + {}^{-}3$ $^{-}2 + {}^{-}3 + {}^{-}2$

$^{-}1 + {}^{-}3 + {}^{-}3$ $^{-}3 + {}^{-}2 + {}^{-}2$

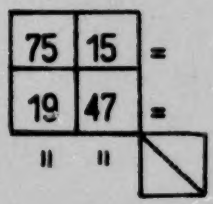
WHAT'S THE SUM?

Sylvia discovered something interesting. Follow the directions below. See if you can find what she discovered.

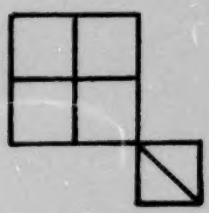
- Add across.
- Add down.
- Add your results, across and down.
- Put your answers in the triangles.

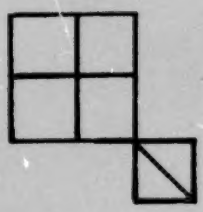


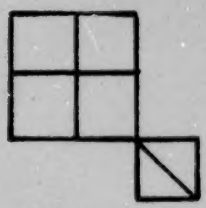
1. Try your luck on this one.



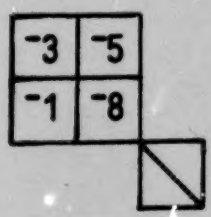
2. Make up some of your own.

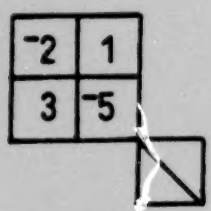
a. 

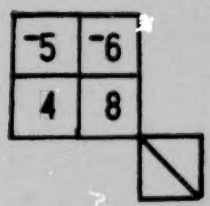
b. 

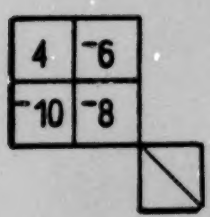
c. 

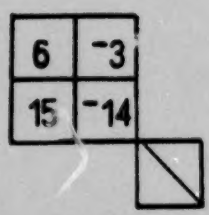
3. Sylvia wondered whether her discovery would work for negative numbers.

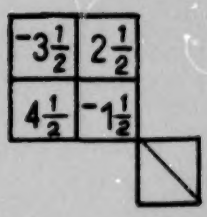
a. 

b. 

c. 

d. 

e. 

f. 

What's The Sum?

Mathematics teaching objectives:

. Practice adding and subtracting with integers using intuitive approaches.

Problem-solving skills pupils might use:

. Look for patterns.
. Invent other problems which can be solved by certain solution procedures.
. Work backwards.

Materials needed:

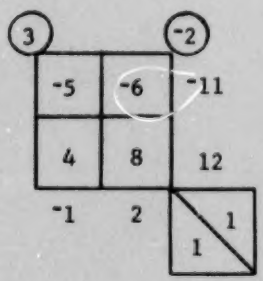
. None

Comments and suggestions:

. The approach used in the lesson is to introduce and then provide experience with the two by two "magic square" using whole numbers followed by ones in which integers are used. See if pupils can make this extension on their own.

. Pupils should be able to work the exercises based upon the two previous lessons. If they are unable to do so you may show them how to do the exercises by relating addition as moves along a number line.

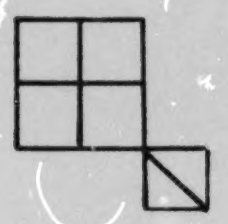
. During culminating discussions you may wish to have pupils share some of their created problems. You might also wish to call pupils' attention to the total of the diagonal sums. This total should be the same as the other totals. e.g.

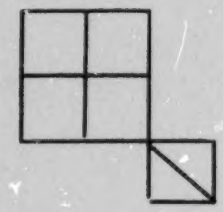


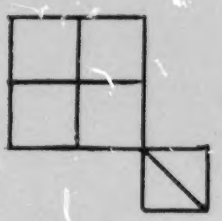
3 + -2 = 1

. You might remind pupils that this is an example of the combined commutative-associative property for addition. Some math educators call this the "rearrangement" property of addition.

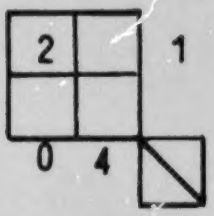
4. Make up numbers. Use both positive and negative

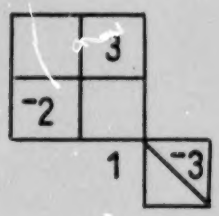
a. 

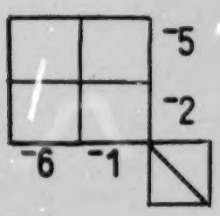
b. 

c. 

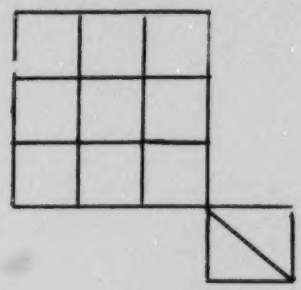
5. Complete the following:

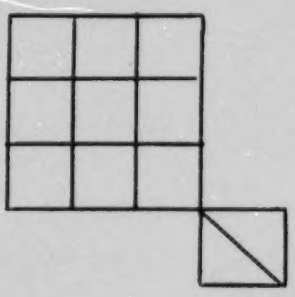
a. 

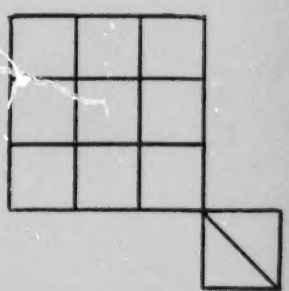
b. 

c. 

6. Will Sylvia's discovery work on a 3 by 3? Try some to see. Use both positive and negative numbers.

a. 

b. 

c. 

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What's The Sum?

Answers:

1.

73	15	= 88
19	47	= 66
92	62	154 / 154

2. Answers will vary.

3. a.

-3	-5	-8
-1	-8	-9
-4	-13	-17 / -17

b.

-2	1	-1
3	-5	-2
1	-4	-3 / -3

c.

-5	-6	-11
4	8	12
-1	2	1 / 1

d.

4	-6	-2
-10	-8	-18
-6	-14	-20 / -20

e.

6	-3	3
15	-14	1
21	-17	4 / 4

f.

$-3\frac{1}{2}$	$2\frac{1}{2}$	-1
$4\frac{1}{2}$	$-1\frac{1}{2}$	3
1	1	2 / 2

4. Answers will vary.

5. a.

2	-1	1
-2	5	3
0	4	4 / 4

b.

-2	3	1
-2	-2	-4
-4	1	-3 / -3

c. Answers will vary. One possibility is given.

-2	-3	-5
-4	2	-2
-6	-1	

6. Yes.
An example is given.

3	5	-1	7
-2	6	5	9
3	-10	2	-5
4	1	6	11 / 11

A TRICKY PATTERN

Study this pattern: 3, 4, 7, 11, 18, 29, 47, 76

Note that $3 + 4 = 7$
$4 + 7 = 11$
$7 + 11 = 18$
etc.

Use the same rule to complete the patterns below.

1. 4, 5, ___, ___, ___, 37

2. 1, 2, ___, ___, ___, ___, 21

3. 2, 1, ___, ___, ___, ___, 18

4. 6, 10, ___, ___, ___, ___, 110

5. 2, ___, 10, ___, ___, ___, 74

6. 3, ___, ___, 7, ___, ___, ___, 50

7. 5, ___, ___, 5, ___, ___, ___, ___, 65

8. 6, ___, ___, 4, ___, ___, ___, 35

9. 8, ___, ___, ___, 7, ___, ___, ___, 41

10. ___, ___, ___, ___, 15, 20

11. ___, ___, ___, ___, ___, ___, ___, ___, 10, 16

12. 10, ___, ___, ___, ___, 0, ___, ___, ___, ___, 10

A Tricky Pattern

Mathematics teaching objective:

- Practice mental computation using an intuitive knowledge of addition with integers.

Problem-solving skills pupils might use:

- Guess and check.
- Look for patterns.

Materials needed:

- None

Comments and suggestions:

- It is best if most of the problems are done in class so that you provide assistance when necessary.
- Give pupils some time to work on their own before giving any instruction. Maybe they will need no assistance! The sequences are rather straight forward through problem 5. Beyond this point pupils may need to use guess and check procedures.
- Problems 8 through 12 involve operations with negative numbers. Problems 10 and 11 imply strongly the "what must I add to this number" interpretation for subtraction. You may wish to capitalize upon this opportunity to discuss subtraction using moves from one point to another on the number line. Do not use the "change the sign and add" rule for subtraction at this time. The objective is to gain intuitive understandings, not to develop rules.
- When the lesson is discussed, encourage pupils to mention the strategies they used to complete the sequences.

Answers:

1. 4, 5, 9, 14, 23, 37
2. 1, 2, 3, 5, 8, 13, 21
3. 2, 1, 3, 4, 7, 11, 18
4. 6, 10, 16, 26, 42, 68, 110
5. 2, 8, 10, 18, 28, 46, 74
6. 3, 2, 5, 7, 12, 19, 31, 50
7. 5, 0, 5, 5, 10, 15, 25, 40, 65
8. 6, -1, 5, 4, 9, 13, 22, 35
9. 8, -3, 5, 2, 7, 9, 16, 25, 41
10. 15, -5, 10, 5, 15, 20
11. 4, -2, 2, 0, 2, 2, 4, 6, 10, 16
12. 10, -6, 4, -2, 2, 0, 2, 2, 4, 6, 10

Grade 8

V. EQUATION SOLVING

V. EQUATION SOLVING

Teachers often are at a loss as to what to do at "open house." One teacher of 8th-grade pupils decided to see how the parents would handle the equation given at the right. The responses were varied.

$$\frac{5n - 9}{6} + 15 = 31$$

"It's been too many years. I can't remember how to begin."

"I never had algebra in school."

"You mean my daughter is solving problems like that...I didn't think she was that good in math."

Within the brief 10-minute session, the problem was solved and in doing so, the teacher was able to emphasize several major objectives in mathematics teaching, including some problem-solving skills.

Pupils (and parents) can solve problems like this one. But they do not need any formal algebraic procedures. Instead, a guess-check-refine strategy is used. In addition to finding a solution, pupils get much practice in computation, in making charts, in looking for patterns, and in rounding.

Some teachers may feel that the problems in the packet are too difficult for pupils at this level. They may be difficult if equation-solving properties are insisted upon. They are not difficult if pupils are allowed to use guessing. And, in the process, pupils will gain a much better understanding of the overall structure of an equation and the meaning of a variable.

Do 8th-grade pupils need to learn equation-solving properties? Probably not. The introduction of these could easily be postponed until the first course of algebra. However, if such properties are introduced at this level, they should come only after pupils have been exposed to some "big" equations and are ready to learn methods that are more efficient than guess and check.

This section should be treated as a unit and done in sequence from beginning to end. Also, for the most part, the lessons should be done in class. Such procedure might avoid the lack of understanding that could result from premature instruction in formal equation-solving procedures by parents or others.

These lessons are algebra readiness activities. Formal equation-solving procedures should be deferred until after this unit.

PUZZLE PROBLEMS

Arthur found this equation in his brother's algebra book:

$15N + 23 = 398$ (Remember, 15N means 15 times N.)

His brother explained that this equation is like a puzzle problem:

I'm thinking of a number.
If you multiply it by 15 and add 23 you get 398.
What is my number?

Class Exercises

1. Arthur's first guess was 12. Was his guess too large or too small?

2. Arthur's next guess was 30. Was the guess too large or too small?

3. What is the solution to $15N + 23 = 398$?

4. Here are other equations Arthur found in the book. First read the equation as a puzzle problem. Then find the solution.

   a. $2N - 4 = 24$

   b. $\frac{M}{5} + 1 = 3$

   c. $\frac{4A + 6}{2} = 7$

   d. $\frac{2N}{3} - 4 = 0$

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Puzzle Problems

Mathematics teaching objectives:

- Translate equations into puzzle problems.
- Solve equations using a guess-and-check technique.
- Compute mentally.
- Create equations which have a given solution.

Problem-solving skills pupils might use:

- Guess and check.
- Record solution possibilities or attempts.
- Work backwards.

Materials needed:

- None

Comments and suggestions:

- Symbols other than "x" are used for variables since pupils are accustomed to using "x" as a multiplication symbol.
- Pupils will need to know that the bar in expressions such as $\frac{M}{5}$ or $\frac{4A + 6}{2}$ can be thought of as a symbol for division.
- Encourage pupils to record their guesses and how far their guesses are off. Do not rush the process.
- The Class Exercises were designed to be total group activities. Exercises 1 and 2 should be done during class time so assistance can be provided if needed. If work is to be done out of class, assign Exercises 3 and 4 on page 133.
- Pupils may discover their own methods for solving equations that do not rely on guessing. After the entire unit is completed, the teacher could call attention to some of the methods pupils have discovered.

Answers:

Class Exercises

1. Too small
2. Too large
3. 25
4. a. 14 b. 10 c. 2 d. 6

Exercises

1. Answers will vary.
2. a. 26 d. 13 g. 1
   b. 15 e. 120 h. 2
   c. 16 f. 20 i. 10
3. Answer will vary. One possibility is $\frac{15N}{10} + 70 = 73$.
4. Answers will vary. A possible equation for each one is given below.
   a. $5(N + 2) = 55$ b. $\frac{25 - N}{4} = 3$
   c. $\frac{25 + N}{5} + 2 = 7$ d. $\frac{4N - 1}{3} = 3$

Exercises

1. Write each equation as a puzzle problem.

   a. $2N + 1 = 53$ b. $\frac{M - 5}{3} = 5$

2. Find the solution to each of the following equations.

   a. $2N + 1 = 53$ d. $9A - 7 = 110$ g. $\frac{12Y}{4} + 2 = 5$

   b. $3T - 1 = 44$ e. $\frac{N}{10} + 2 = 14$ h. $\frac{7P - 2}{4} = 3$

   c. $5B + 120 = 200$ f. $\frac{M - 5}{3} = 5$ i. $\frac{2N + 1}{3} - 3 = 4$

3. In each of the exercises above we searched for a solution that was "hidden" in the equation.

   For example, in exercise 2h, the hidden solution is 2.

   Here's another equation in which the hidden solution is 2:

   $3(17N - 30) = 12$.

   Write a different equation in which the hidden solution is 2.

4. Create equations which have the following hidden solutions.

   a. 9 b. 13 c. 0 d. $2\frac{1}{2}$

THE HIDDEN SOLUTION

Tammy made up this equation in which the solution is hidden:

$6N - 2 = 48$.

Tammy's classmates were stumped. Can you figure out why?

Class Exercises

1. The chart at the right will help you narrow in on the solution to Tammy's equation.

   a. Complete the chart.

   b. What whole number is the solution closest to?

Guess	Result	Too large or too small?
5	28	Too small
11	?	?
9	?	?
8	?	?

2. Each of the following has a fraction as a solution. What whole number is each solution closest to? Use a chart to help you decide.

   a. $3(N - 11) = 14$ b. $\frac{5B + 2}{3} = 13$

Exercises

Find each solution. If it is fractional, find the closest whole number. Use a chart if necessary.

1. $3Y + 7 = 20$
2. $4N - 1 = 30$
3. $5R + 13 = 57$
4. $6P + 8 = 50$
5. $\frac{3N}{2} = 49$
6. $\frac{4N + 1}{2} = 10$
7. $\frac{5R - 3}{3} + 2 = 15$
8. $4(3N + 15) = 100$

9. Create an equation in which the hidden solution is
   a. between 4 and 5. b. closer to 8 than 9.

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The Hidden Solution

Mathematics teaching objectives:

- Mentally compute with whole numbers and fractions.
- Informally solve equations.
- Round numbers.

Problem-solving skills pupils might use:

- Guess and check.
- Make and use a table.
- Look for patterns.
- Determine limits.

Materials needed:

- None

Comments and suggestions:

- Do the Class Exercises as a total group activity, allowing pupils to tell how they know which is the closest whole number answer. Be certain that charts are made for each of the equations in the Class Exercises.
- Allow sufficient class time for pupils to solve most of the Exercises in class. The extension could be given as homework.

Answers:

Class Exercises

1. a.

Guess	Result	Too large or too small?
5	28	Too small
11	64	Too large
9	52	Too large
8	46	Too small

b. 8

2. a. 16 b. 7

Exercises

1. 4
2. 8
3. 9
4. 7
5. 33
6. 5
7. 8
8. 3

9. Answers will vary. A possible equation for each part is given below.

a. $\frac{2N + 3}{4} = 3$ b. $\frac{3N + 2}{3} = 9$

ARTHUR'S DILEMMA

Can you help out Arthur? He's stumped on this equation:

$$2N + 3N = 85$$

Arthur's brother told him that he must replace each N with the same number. But Arthur is still stumped.
He tried 35 and that was way too large. See if you can help him out.

Exercises

1. What is the solution to the equation that stumped Arthur?

2. We can read the equation as a puzzle problem:

   "I'm thinking of a number.
   Twice the number added to three times the
   same number gives a result of 85.
   What is my number?"

   Write the following equation as a puzzle problem: $\frac{5N}{2} + 3N - 2 = 9$.

3. Solve the following equations. If a solution is a fraction, find the closest whole number.

   a. $4A + 2A = 42$

   b. $7B - 3B = 80$

   c. $2C + 5C = 62$

   d. $\frac{5D}{2} + 3D - 2 = 9$

   e. $\frac{E + 5}{2} + E = 19$

   f. $5(F + 1) - 2F = 17$

   g. $3(G + 4) + 3(G - 4) = 50$

   h. $H + H + H = 50$ (Each H must be replaced with the same number.)

   i. $5I - 2I + 4I = 100$

EXTENSIONS

Solve for N.

a. $\frac{4N + 25}{8} + 2N = 100$

b. $\frac{N}{10} + \frac{10}{N} = 10$

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Arthur's Dilemma

Mathematics teaching objectives:

- Mentally compute with whole numbers and fractions.
- Read and interpret equations.
- Round numbers.
- Informally solve equations involving like terms.

Problem-solving skills pupils might use:

- Guess and check.
- Record guesses or solution attempts.

Materials needed:

- None

Comments and suggestions:

- It may help some pupils if equations are first translated into equivalent English statements. For example, 2N + 3N = 85 could become "Twice the number added to three times that number is 85."
- Encourage pupils to discuss shorter ways to solve equations by economizing on their guesses. Again, this is not the time for the teacher to direct the class toward a common method. If this is done at all, it should come at the end of the unit.

Answers:

Exercises

1. 17
2. Answers will vary.
3. a. 7
   b. 20
   c. 9 is closest
   d. 2
   e. 11
   f. 4
   g. 8 is closest.
   h. 17 is closest.
   i. 14 is closest.

Extensions

a. 39 is closest.

b. Both 100 and 1 are good approximations. Each gives a result of $10\frac{1}{10}$. An even better approximation is 99.

ANOTHER DILEMMA

$2N + 30 = 6N - 2$

"2N on one side,
6N on the other;
What does it all mean?
I guess I'll ask by brother."

Class Exercises

1. What do you think was bothering Arthur this time?

2. How is this new equation different from those you have been solving?

Arthur's brother explained that each letter still must be replaced with the same number. He also suggested that a chart be used.

3. Study the chart.

N	Left Side	Right Side
10	50	58
11	52	64
12	54	70

   a. Are we getting closer to the solution or further away?

   b. Add more numbers to the chart. What is the solution to the equation?

4. Use a chart to help you solve $5N + 3 = 2N + 15$.

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Another Dilemma

Mathematics teaching objectives:

- Use informal procedures for solving single variable equations with a variable on both sides of the equation.
- Mentally compute with whole numbers.

Problem-solving skills pupils might use:

- Make and use a table.
- Look for patterns and make predictions.
- Guess and check.

Materials needed:

- None

Comments and suggestions:

- Use the Class Exercises for developing the "chart" procedure for arriving at solutions. Emphasize that carefully kept records of guesses in a table can bring the solution process under control. Pupils need to realize that some experimentation with different values for the variable is usually necessary before trends can be observed which will lead to a solution.
- Allow pupils sufficient time to solve these equations independently and in an organized way even though they may only solve three or four of the equations during the supervised study period.
- You may need to remind the pupils that the process they use to get the answer is more important for developing problem-solving ability than is the answer itself.
- Have pupils share and explain their problem-solving procedures with their classmates. Also, have them share their created "monsters" with the class.

Answers:

Class Exercises

1. Probably because the Ns appear on both sides of the equation.
2. Letters are on both sides of the equation.
3. a. Further away b. 8
4. 4

Exercises

1. 1 2. 3 3. 21 4. 8 5. 5 6. 0

Extensions

a. 4 b. Answers will vary. Pupils may come up with equations such as $\frac{2N + 2}{2} = N$ (no solution) or $3(n + 2) = 3n + 6$ (infinitely many solutions). If they do, the equation would serve as a "springboard" into the next lesson.

Exercises

Use a chart to help you. Solve the following equations. Remember, in each case you must replace each letter with the same number.

1. $7R + 1 = 6R + 2$
2. $Y + 2 = 2Y - 1$
3. $4N = 2N + 42$
4. $\frac{N}{2} + 3 = N - 1$
5. $3(P + 1) = P + 13$
6. $5T + 2 = 3T + 2$

EXTENSIONS

a. Here's a "monster" that Arthur found. See if you can solve it.

$$\frac{4(2N + 16)}{3} = N + 28$$

b. Create a "monster" of your own. Be sure you know the solution before giving it to a classmate to solve.

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UNUSUAL EQUATIONS

Tammy is always coming up with unusual problems.

Why do you think the equations she created at the right are so unusual?

Solve for M.
$M - 2 = M + 3$

Solve for N.
$5(N + 4) = 5N + 20$

Exercises

If possible, find the solution to each equation below. Two equations have no solutions; two have infinitely many solutions. If no solution, write "no;" if infinitely many, write "many."

1. $3N + 6 = 21$
2. $\frac{2M + 1}{7} = 5$
3. $5P + 6 = 2P + 45$ ← (Remember, a chart is helpful when a letter occurs on both sides of the equation.)
4. $2R - 3 = 2R + 1$
5. $6N + 18 = 6(N + 3)$
6. $2B + (5 + 3) = 8 + 2B$
7. $\frac{2T + 4}{4} - 1 = 23$
8. $\frac{N}{N + 1} = 1$

EXTENSIONS

a. Create an equation which has no solution.

b. Create an equation which has many solutions.

Unusual Equations

Mathematics teaching objectives:

- Introduce equations which have more than one solution.
- Introduce equations which have no solutions.

Problem-solving skills pupils might use:

- Identify problems in which no solution is possible.
- Identify problems in which more than one solution is possible.
- Guess and check.

Materials needed:

- None

Comments and suggestions:

- Use the equations in the "bubble" as the basis for introducing the lesson to the entire class.
- Suggest that pupils translate equations (for example: $M - 2 = M + 3$, $2R - 3 = 2R + 1$, and $2B + (5 + 3) = 8 + 2B$) into English statements. Such translations may reveal that there are no possible solutions or infinitely many solutions.
- Discuss with the class the meaning of "infinitely many" solutions.
- If this lesson is the last one on equation solving, tell the pupils they will be learning more systematic procedures for solving equations next year in algebra.
- During the culminating discussion, have pupils share their created equations and the "rules" they used in inventing them.

Answers:

1. 5
2. 17
3. 13
4. No
5. Many (This is an example of the distributive property.)
6. Many (This is an example of the commutative property of addition.)
7. 46
8. No (This is a good discussion item. Note that the denominator is always one more than the numerator. Thus, the fraction will never be equal to 1 no matter how large the replacement for N.)

Extensions

Answers will vary.

Grade 8

VI. PROTRACTOR EXPERIMENTS

VI. PROTRACTOR EXPERIMENTS

Four investigations are provided which rely on measuring with a protractor. The measurements taken are used to make a conjecture. Several more examples are then tried to verify the conjecture. Repeating an experiment many times and looking for patterns is an example of an inductive approach. This approach, of course, is the way much mathematics has been discovered.

Each investigation is also explained using a deductive approach. Facts which have been previously proven or agreed upon are used to verify or prove a new statement. Although our proofs are not rigorous, enough is given so teachers can "convince" pupils the statement is true. How much of this deductive approach is used is left to the individual teacher. If a deductive approach is used, the intent should be to convince pupils of the truth of the statement, and to show how a conclusion can be reached without relying on measurement or experimentation.

To be successful with this unit, pupils must have some previous experience measuring angles and reading a protractor. Some of the sides of the angles may need to be extended so that pupils can more easily use their protractors. These investigations should be done in sequence.

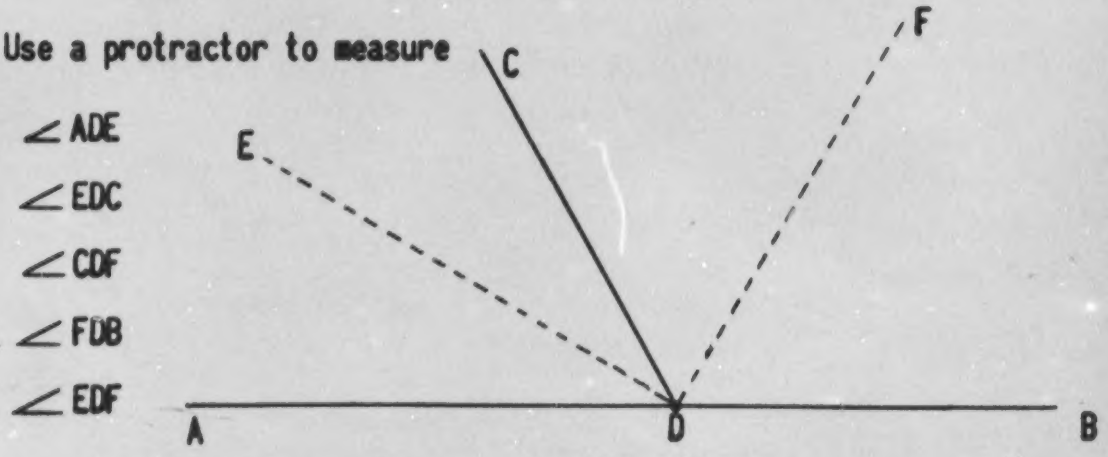
Be sure to comment about the types of errors that can occur when using a protractor:

- . not lining up the protractor accurately.
- . reading the scale from the wrong direction.
- . reading the scale inaccurately.
- . using a protractor that was not accurately manufactured.
- . making a reading error due to the calibration of the protractor.

PROTRACTOR PRACTICE

Do your work carefully. Otherwise, patterns may not be apparent.

1. Use a protractor to measure



Place your measurements in the first row of this table. The rest of the table will be filled in later.

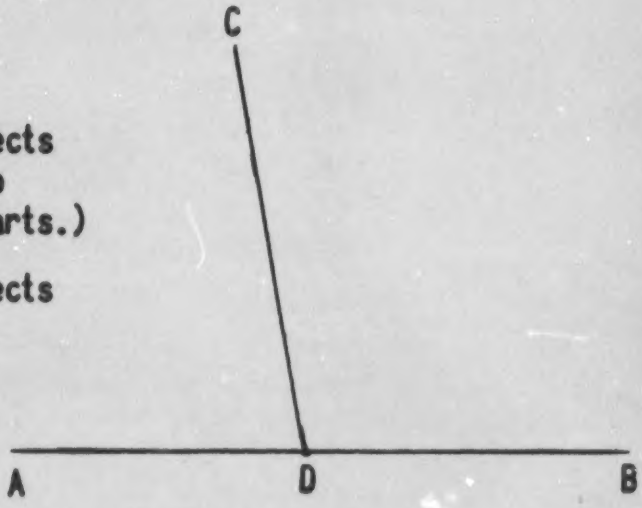
∠ADE	∠EDC	∠CDF	∠FDB	∠EDF

2. Use a protractor.

   a. Draw a line ED that bisects ∠ADC. (Bisect means to divide into two equal parts.)

   b. Draw a line FD that bisects ∠CDB.

   c. Measure the resulting angles and place your measurements in the second row of the table.



Protractor Practice

Mathematics teaching objectives:

- Find properties of angle bisectors.
- Use a protractor.

Problem-solving skills pupils might use:

- Make and use a drawing.
- Look for patterns.
- Make predictions based upon data.

Materials needed:

- Protractor

Comments and suggestions:

- A reminder to be accurate is appropriate. Patterns may not be apparent if measurements are several degrees off.
- Pupils should know that experimentation and measurement often take a lot of effort. But many discoveries in mathematics, and other areas, are made in this way.
- A deductive approach to this experiment can be summarized in the following way:
  - AB is a straight line, so $\angle ADC + \angle CDB = 180^\circ$.
  - Segment DE bisects $\angle ADC$ and segment DF bisects $\angle CDB$.
  - Half of $\angle ADC$ (or $\angle EDC$) + half of $\angle CDB$ (or $\angle CDF$) = $\frac{1}{2}$ of 180 or 90°. In other words, $\angle EDC + \angle CDF = \angle EDF = 90^\circ$.

Answers:

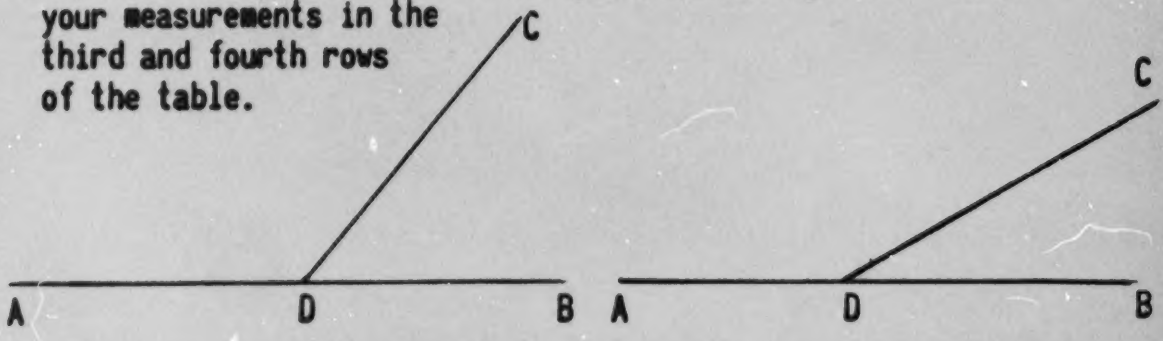
Pupil answers may vary slightly.

1.

$\angle ADE$	$\angle EDC$	$\angle CDF$	$\angle FDB$	$\angle EDF$
30	30	60	60	90
40	40	50	50	90
65	65	25	25	90
75	75	15	15	90

2. See table above.
3. See table above.
4. $\angle EDF$ should always be close to 90°.
5. a. 112° b. 56° c. Answers will vary.
6. a. 60° b. 30° c. 75° d. 90°

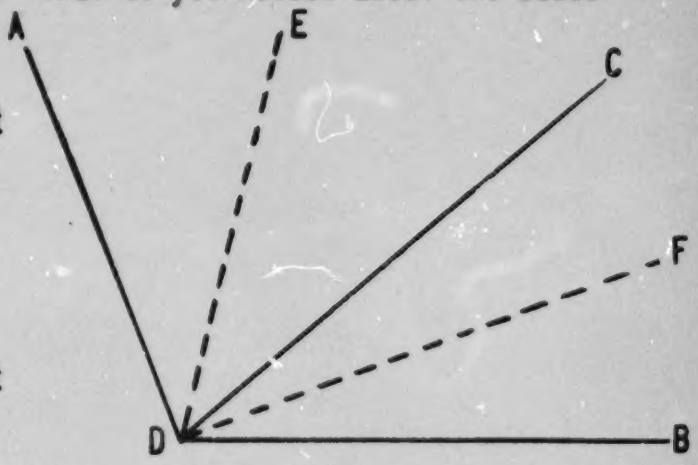
3. Repeat the experiment above using these two figures. Place your measurements in the third and fourth rows of the table.



4. Study the completed table. What do you notice about the sizes of $\angle EDF$?

5. In the figure at the right

   ED bisects $\angle ADC$
   FD bisects $\angle CDB$



   a. Use a protractor to measure $\angle ADB$.

   b. Use reasoning. Do not use a protractor. What should be the measure of $\angle EDF$?

   c. Now measure $\angle EDF$. How close was your prediction?

6. Consider other figures like the one in problem 5. See if you can determine the following answers without relying on measurement.

   a. If $\angle ADB$ is 120°, what is the size of $\angle EDF$?

   b. If $\angle ADB$ is 60°, what is the size of $\angle EDF$?

   c. If $\angle ADB$ is 150°, what is the size of $\angle EDF$?

   d. If $\angle EDF$ is 45°, what is the size of $\angle ADB$?

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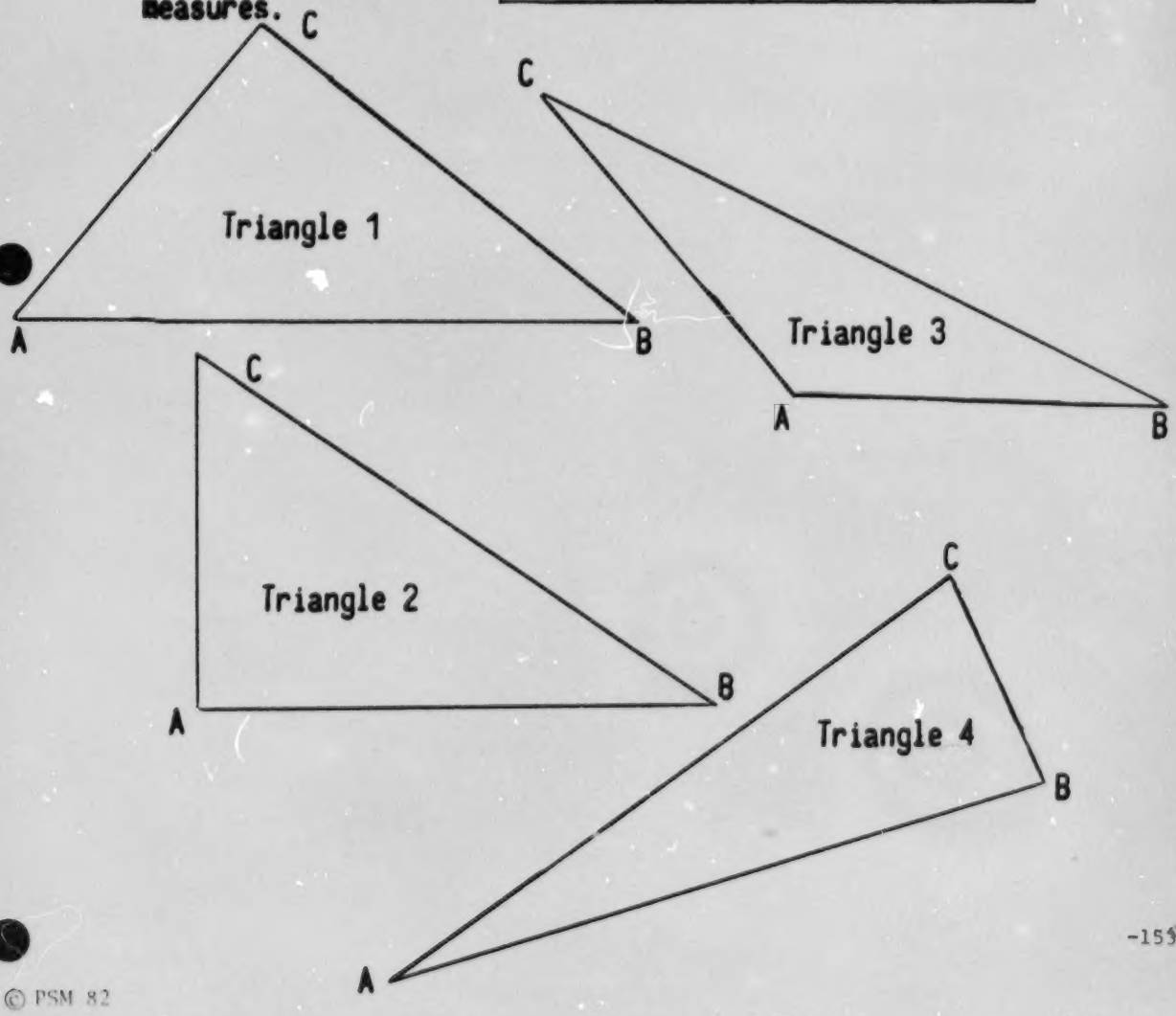
MORE PROTRACTOR PRACTICE

People argue about the strangest things!

Judy and Beth were arguing about the degrees in a triangle. Judy was sure that the angle sum was 200°. Beth was certain that it was not that much, but couldn't remember the exact total.

1. Measure the angles in each triangle. Place your measurements in the table. Then, for each triangle, find the sum of the three measures.

Triangle	A	B	C	Sum
1				
2				
3				
4				



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More Protractor Practice

Mathematics teaching objectives:

- Find interior angle sums of triangles and quadrilaterals.
- Use a protractor.

Problem-solving skills pupils might use:

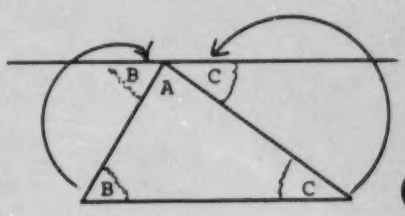
- Make necessary measurements for obtaining a solution.
- Look for patterns.
- Make predictions based upon data.

Materials needed:

- Protractor

Comments and suggestions:

- A reminder to be accurate is appropriate. Patterns may not be apparent if measurements are several degrees off.
- Many eighth-graders are surprised that the sum of the interior angles is always the same.
- An informal deductive approach can be used to verify or prove the angle sum statement. Probably the most convincing way is to tear off $\angle B$ and $\angle C$ and place them as indicated in the drawing. The three angles are pieced together to make a straight line. Therefore, $\angle A + \angle B + \angle C = 180^\circ$.



Answers:

Pupil answers may vary slightly.

1.

Triangle	$\angle A$	$\angle B$	$\angle C$	Sum
1	50	38	92	180
2	90	35	55	180
3	131	25	24	180
4	19	82	79	180

2. a. 100° b. Answers will vary.
3. a. 80° b. Answers will vary.
4. a. Each angle is 90°; the sum of all four is 360°.
   b. Answers will vary. The sum should be close to 360°. To verify the angle sum statement, cut out a quadrilateral. Tear off the four corners. Piece the corners together around a point in "jig saw fashion." All the space around the point should be completely filled. Therefore, the sum of the four angles = 360°.

More Protractor Practice (cont.)

2. Draw a triangle ($\triangle ABC$) with $\angle A = 60^\circ$ and $\angle B = 20^\circ$.

   a. Use reasoning. What do you think the measure of $\angle C$ should be?

   b. Now measure $\angle C$. How close was your prediction?

3. Draw a triangle ($\triangle DEF$) with $\angle D = 45^\circ$ and $\angle E = 55^\circ$.

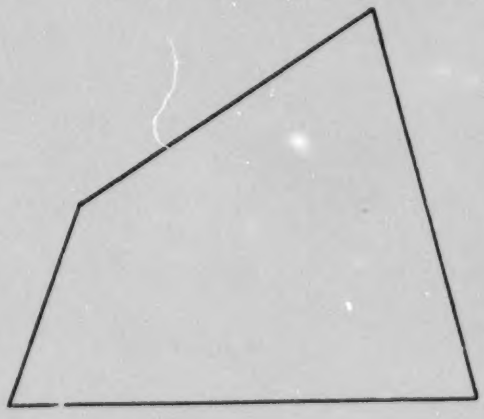
   a. Use reasoning. What do you think the measure of $\angle F$ should be?

   b. Now measure $\angle F$. How close was your prediction?

4. Judy drew a rectangle like the one pictured.

   a. How could she tell that the angle sum is 360° without measuring?

   b. Beth was certain all 4-sided figures would have the same angle sum. Find the angle sum of each 4-sided figure below. Do you agree with Beth?



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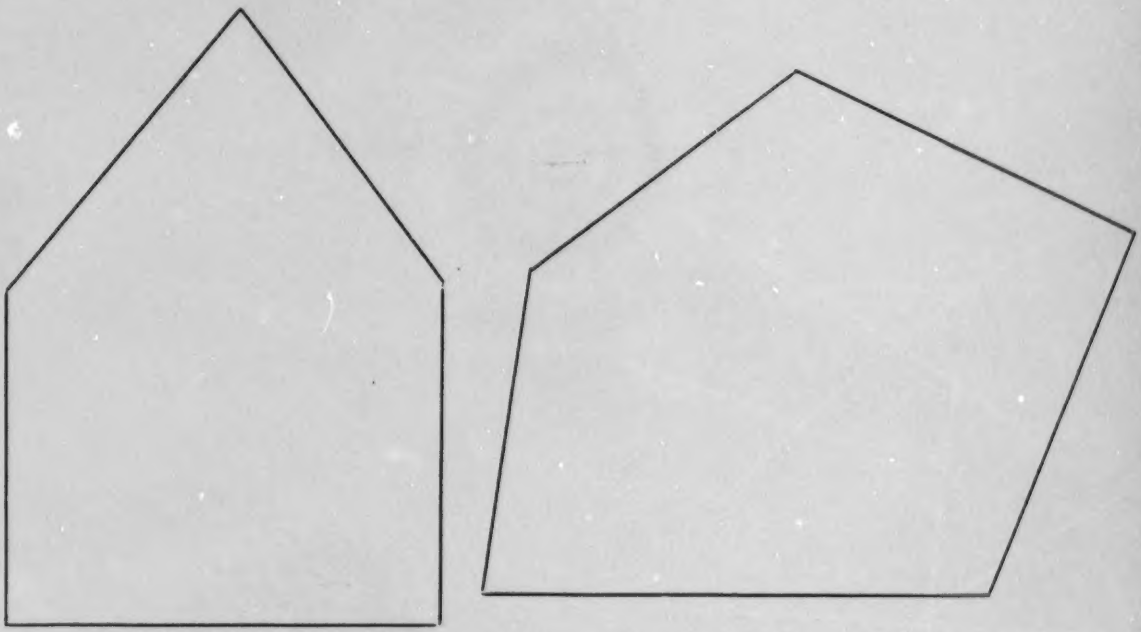
WHAT'S THE SUM?

THE SUM OF THE ANGLES OF A TRIANGLE IS 180°.
THE SUM OF THE ANGLES OF A QUADRILATERAL (4 SIDES) IS 360°.

1. A pentagon is a <u>5-sided</u> figure.
   a. What do you think the angle sum of a pentagon should be?
   b. Find the angle sum of each pentagon drawn below.



   c. How close were your results to the prediction you made?

2. A <u>hexagon</u> is a 6-sided figure.
   a. What do you think the angle sum of a hexagon should be?
   b. Find the angle sum of each hexagon drawn on the next page.

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What's The Sum?

Mathematics teaching objectives:

- Find interior angle sums of polygons.
- Use a protractor.

Problem-solving skills pupils might use:

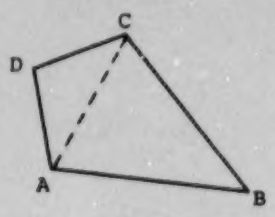
- Make necessary measurements for obtaining a solution.
- Look for patterns.
- Make predictions based upon data.

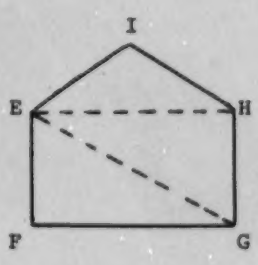
Materials needed:

- Protractor

Comments and suggestions:

- A reminder to exercise care when measuring the angles is appropriate. Patterns may not be apparent if measurements are several degrees off.
- A deductive approach to this activity involves dividing each polygon into triangles. This procedure shows that the quadrilateral below is made up of two triangles. Thus, the sum of the interior angles is 360°. The pentagon below is made up of three triangles. Thus, the sum of the interior angles is 540°.





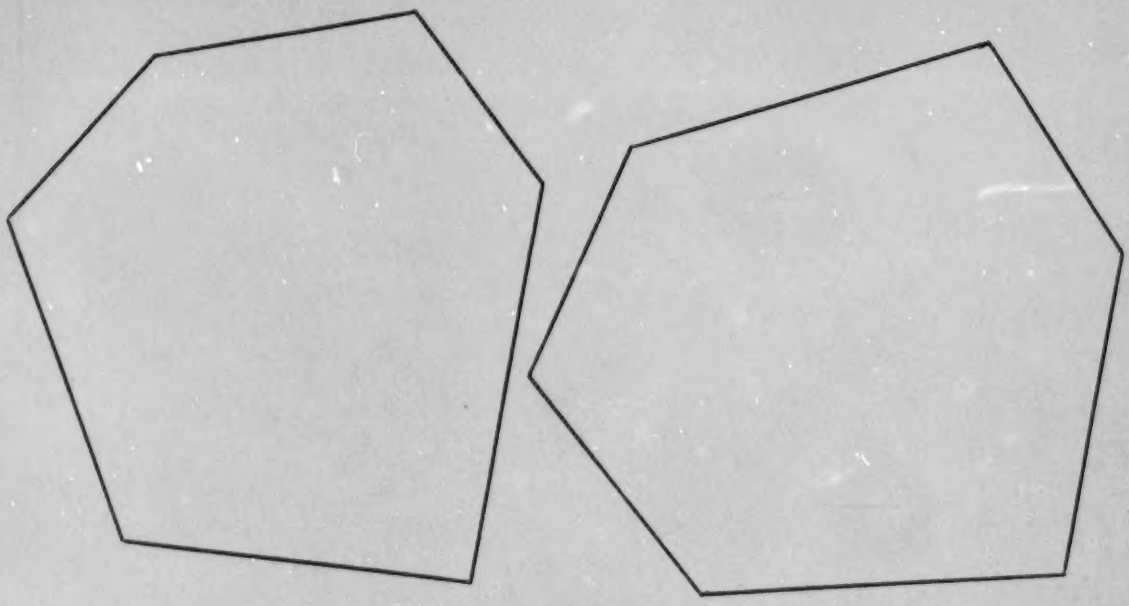
Answers:

1. a. Answers will vary.
   b. The sum should be close to 540°.
   c. Answers will vary.
2. a. Answers will vary.
   b. The sum should be close to 720°.
   c. Answers will vary.

Extension: $17{,}640^\circ$ (The polygon can be subdivided into 98 triangles. Thus, the sum of the interior angles is $98 \times 180^\circ$ or $17{,}640^\circ$.)

3.

No. of sides	Sum of the Angles	
3	180	(1 x 180)
4	360	(2 x 180)
5	540	(3 x 180)
6	720	(4 x 180)
7	900	(5 x 180)
8	1080	(6 x 180)
9	1260	(7 x 180)
10	1440	(8 x 180)



c. How close were your results to the prediction you made?

3. Complete this chart:

Number of Sides	Sum of the Angles
3	180
4	360
5	540
6	
7	
8	
9	
10	

EXTENSION

What is the angle sum of a 100-sided figure?

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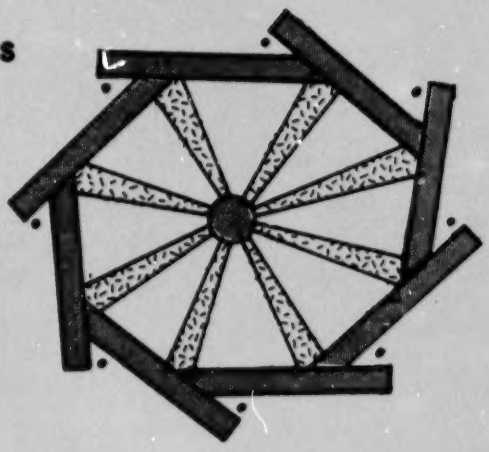
WATER-WHEEL GEOMETRY



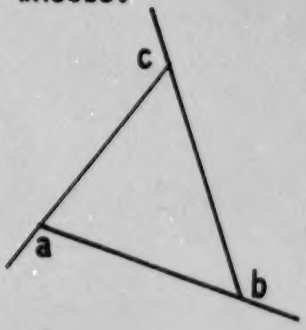
The following experiment involves exterior angles. A simplified picture of a water-wheel will help you understand what to do.

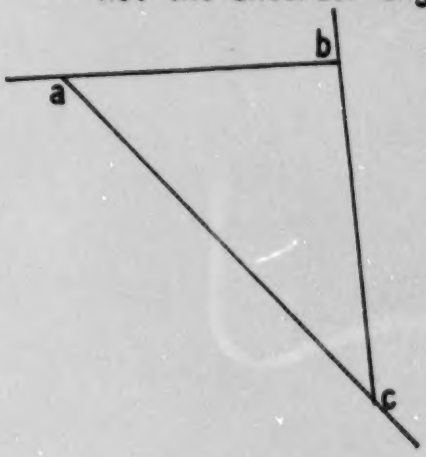
The angles with the dots inside are called <u>exterior angles</u>.

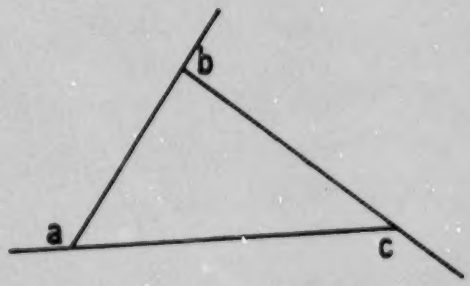
In the previous lesson you discovered a pattern involving the sum of the inside angles of figures; for example, 180^0, 360^0, 540^0, 720^0, ...
Is there a pattern involving the exterior angles? Let's find out.

1. It's best to begin with simple "water-wheels."

   a. For each figure, measure the exterior angles and find the sum. Record the measurements and the sums. Remember to measure the exterior angles, not the interior angles.







   b. What seems to be the exterior-angle sum of a triangle?

Water Wheel Geometry

Mathematics teaching objectives:

- Find exterior angle sums of polygons.
- Use a protractor.

Problem-solving skills pupils might use:

- Make necessary measurements for obtaining a solution.
- Look for patterns.
- Make predictions based upon data.

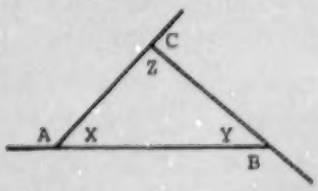
Materials needed:

- Protractor

Comments and suggestions:

- A reminder to be accurate is appropriate. Patterns may not be apparent if measurements are several degrees off.
- Pupils will need to extend the sides of the angles to make the needed measurements.
- Often you can conclude something in geometry without relying on measurement. In the water-wheel problem we can show, by deduction, that the sum must be 360°.

$\angle A + \angle X = 180^\circ$
$\angle B + \angle Y = 180^\circ$
$\angle C + \angle Z = 180^\circ$



Therefore,

$(\angle A + \angle X) + (\angle B + \angle Y) + (\angle C + \angle Z) = 540^\circ$

but $\angle X + \angle Y + \angle Z = 180^\circ$

Therefore, $\angle A + \angle B + \angle C = 360^\circ$

The same argument can be used on other polygons as well.

- Pupils are surprised that the angle sum for exterior angles of a triangle is the same as the angle sum of the exterior angles of any polygon.

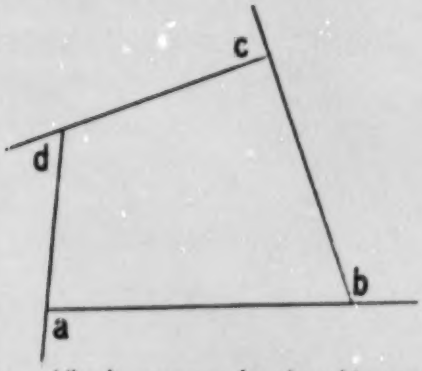
Answers:

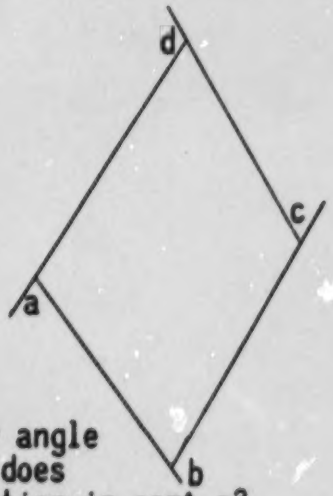
1. b. The exterior angle sum should be close to 360°.
2. a. Answers will vary.
   b. The exterior angle sum should be close to 360°.
   c. Answers will vary.
3. a. Answers will vary
   b. The exterior angle sum should be close to 360°.
4. a. 360° b. 360° c. 360°

2. Remember, a quadrilateral is a 4-sided figure.

   a. Make a prediction about the exterior-angle sum of a quadrilateral.

   b. Find the exterior-angle sum of each quadrilateral drawn below.



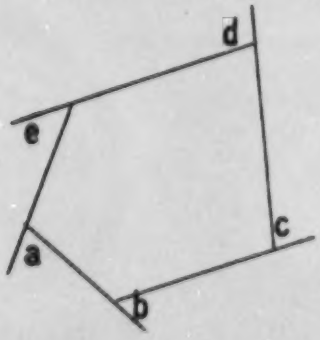


   c. What seems to be the exterior angle sum of a quadrilateral? How does this compare with your prediction in part a?

3. Remember, a pentagon is a 5-sided figure.

   a. Make a prediction about the exterior-angle sum of a pentagon.

   b. Find the exterior-angle sum of the pentagon drawn below. How does this sum compare with your prediction in part a?



4. What would you predict to be the exterior-angle sum of a

   a. 10-sided figure? b. 100-sided figure?

   c. 1000-sided figure? (You don't have to draw these to check your predictions unless you want to.)

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Grade 8

VII. INVESTIGATIONS IN GEOMETRY

VII. INVESTIGATIONS IN GEOMETRY

An eighth-grade course in mathematics traditionally serves several purposes: (a) review and maintenance of basic skills, (b) introduction of new topics such as statistics or straightedge-compass constructions, and (c) study of algebra-readiness topics. An eighth-grade course also should include some geometry-readiness topics. The ones included in this section focus on quadrilaterals, scale drawings, congruence properties, and the Pythagorean Property. Other topics that could be included are similarity properties, parallel properties, and circle properties.

These activities use an inductive approach. An investigation is repeated several times to gather data. The data then are analyzed to form a conjecture. Sometimes the investigation is repeated to confirm the conjecture. This approach differs from the deductive approach where pupils use previously proven facts to verify a conjecture.

The seven geometric investigations given here are designed for pupils to discover certain basic geometric properties. If mastery is the goal, additional practice and applications will be needed.

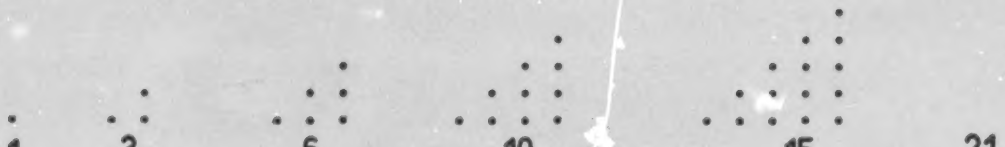
The lessons in this section can be worked in any order except "Quadrilaterals" which should be worked before "Snipping Quadrilaterals." Also, "Pythagoras" may be better used before "Oil Wells," but only if you intend to have pupils use the Pythagorean Property to help solve the problems in "Oil Wells."

Special materials needed for these lessons include:

- metric rulers
- protractors
- compasses
- scissors.

PATTERNS IN GEOMETRY

1.



a. Show how to use 21 dots to form a <u>triangular</u> pattern.

b. Many problems in geometry can be solved by making accurate drawings and then looking for number patterns. Here is the triangular number pattern: 1,3,6,10,15,21,... Extend the pattern. Is 78 a triangular number?

2. These numbers are called <u>rectangular</u>.

a. Explain why these are often called rectangular numbers?

b. How many dots are there along each dimension?

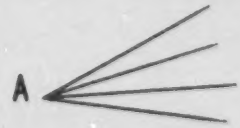
c. What are the next five rectangular numbers?

d. What is the 100th rectangular number?

3. The first figure at the right contains one angle.

The next figure has an extra line. It contains 3 angles--the two small ones and the larger one.

a. Suppose you add another line. Now how many angles are there?



b. How many angles would result if you drew still another line from point A?

c. How many angles would result if you were to draw 13 lines from point A? Try to solve this without actually making a drawing.

-169-

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Patterns In Geometry

Mathematics teaching objectives:

- Find triangular numbers.
- Find patterns in tables of data.

Problem-solving skills pupils might use:

- Make and use a drawing.
- Look for patterns.
- Solve an easier but related problem.

Materials needed:

- None

Comments and suggestions:

- Three different situations that give the same pattern and two similar situations that give different patterns are presented in this activity. Instead of giving all five at the same time, each one could be presented on consecutive days. A summary discussion could then tie the situations together.
- Ask pupils to identify triangle number patterns, e.g., the difference between terms increases by one when the numbers are arranged in sequential order:

  1 3 6 10 15 21 28 ...
  2 3 4 5 6 7

  The pattern can be used to determine in a more convenient way the next triangular number.

Answers:

1. a.

b. Yes, 78 is the twelfth triangular number.

2. a. 42, 56, 72, 90, 110 b. 100 x 101 = 10,100

3. a. 6 b. 10 c. 78

4. a.

sides	4	5	6	7	8	9
triangles	2	3	4	5	6	7

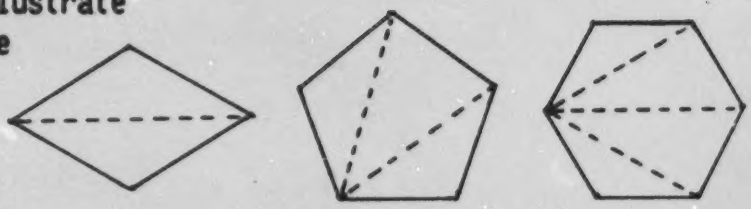
b. 98

5. a.

1 3 6 10

b. 45

4. These drawings illustrate how figures can be subdivided into triangles.

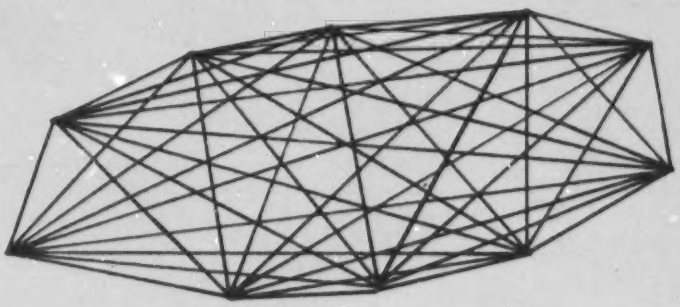


   a. Use this same technique for other figures with 7, 8, and 9 sides. Complete the table.

Number of sides	4	5	6	7	8	9
Number of triangles	2	3	4			

   b. Study the completed table. Look for patterns. If you used the same technique, how many triangles would you expect in a 100-sided figure?

5. Ten very talkative teenagers decided to install their own special telephone system. The system would connect each house with the other nine. How many wires would the system need?



Marty made a drawing. But that didn't help much. There were just too many lines to keep track of.

Kyle decided instead to try simpler cases and look for a pattern.

   a. Make drawings to show the number of wires needed for

      2 persons 3 persons 4 persons 5 persons.

   b. Look for a pattern. How many wires would be needed for 10 persons?

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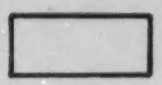
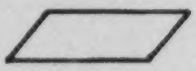
QUADRILATERALS

Square



Rectangle

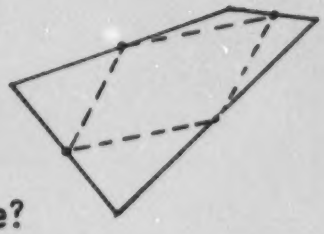


Parallelogram

Diamond

Needed: A millimetre ruler.

1. A quadrilateral is a four-sided figure.
   a. Find the middle point of each side of each quadrilateral on the next page.
   b. Connect the midpoints of the sides as suggested by the figure at the right.
   c. What seems to be the resulting "inside" figure in each case?



2. Try to draw a quadrilateral in which the resulting "inside" figure is (a) a square, (b) a rectangle (but not a square, (c) a diamond (but not a square), (d) none of the figures above.

3. The perimeter of a figure is the distance around.
   a. Find the perimeter of each of the "inside" quadrilaterals on the next page. Place your results in the first column of the chart.

	Perimeter	Diagonal AC	Diagonal BD
I.			
II.			
III.			
IV.			

   b. A diagonal of a quadrilateral is a line joining opposite corners. Measure the diagonals (AC & BD) of each figure on the next page. Record your results in the chart.

4. Study the completed chart.
   a. Write about any observations you've made.
   b. Draw a quadrilateral of your own. Check your observations by making the necessary measurements.

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Quadrilaterals

Mathematics teaching objectives:

- Measure line segments in millimetres.
- Identify different types of quadrilaterals.

Problem-solving skills pupils might use:

- Make necessary measurements for obtaining a solution.
- Look for patterns and/or properties.
- Make generalizations based upon data.

Materials needed:

- Millimetre ruler.

Comments and suggestions:

- Some work with the identification and the classification of quadrilaterals should come prior to this lesson.
- Encourage pupils to be accurate with their measurements. The final generalization will be difficult to make if measurements are several millimetres off.

Answers:

1. a, b. Pupils' drawings

   c. Parallelogram

2. a. Start with a quadrilateral with perpendicular and equal diagonals. (Most pupils will begin with a square, but this is not necessary.)

   b. Start with a quadrilateral with perpendicular and unequal diagonals. (Most pupils will begin with a diamond.)

   c. Start with a quadrilateral with non-perpendicular but equal diagonals. (Most pupils will begin with a rectangle.)

   d. All "inside figures" will be parallelograms.

3.

	Perimeter	Diagonal AC	Diagonal BD
I.	160	100	60
II.	180	120	60
III.	170	70	100
IV.	180	80	100

4. a. The perimeter is the same as the sum of the diagonals.

   b. Pupils' drawings

Quadrilaterals (cont.)

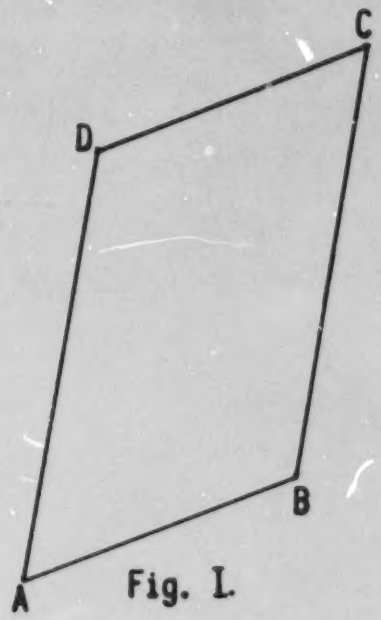

Fig. I.

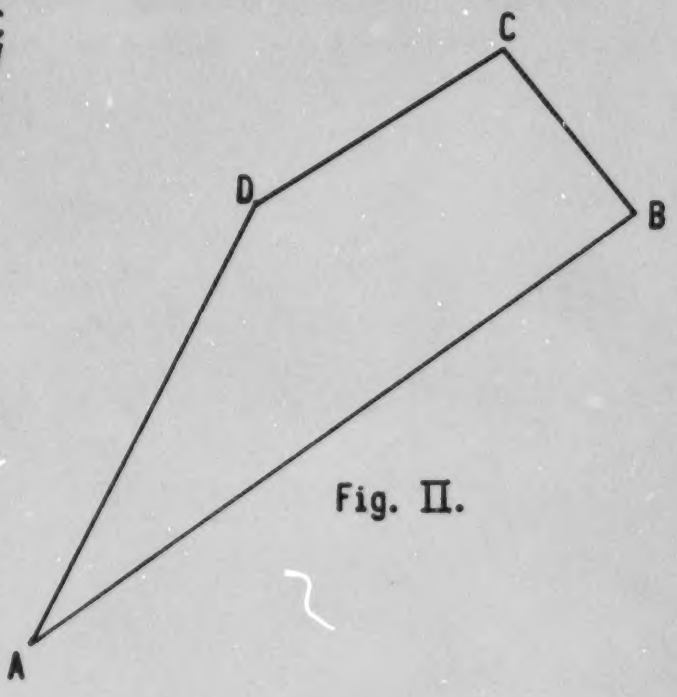

Fig. II.

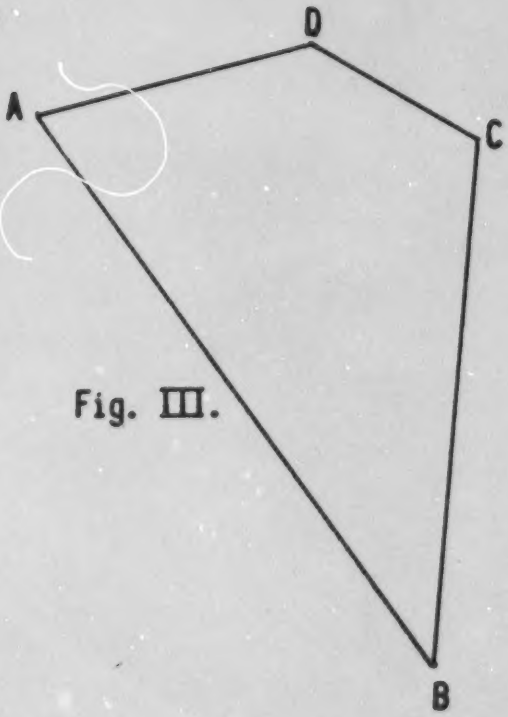

Fig. III.

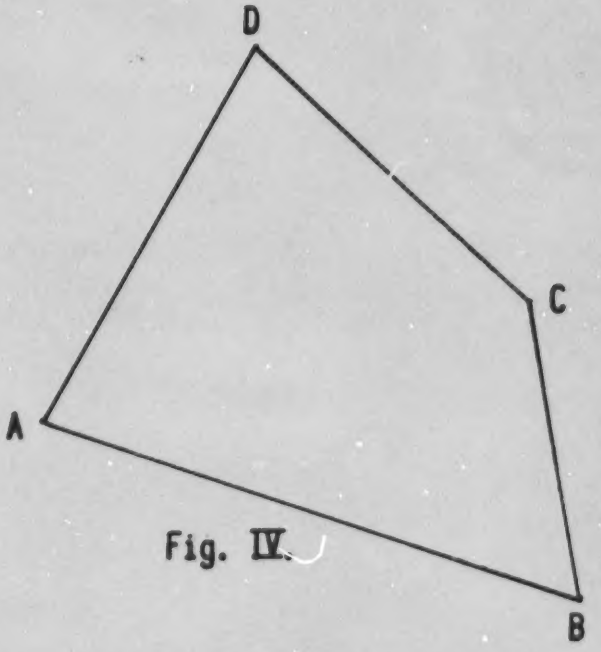

Fig. IV.

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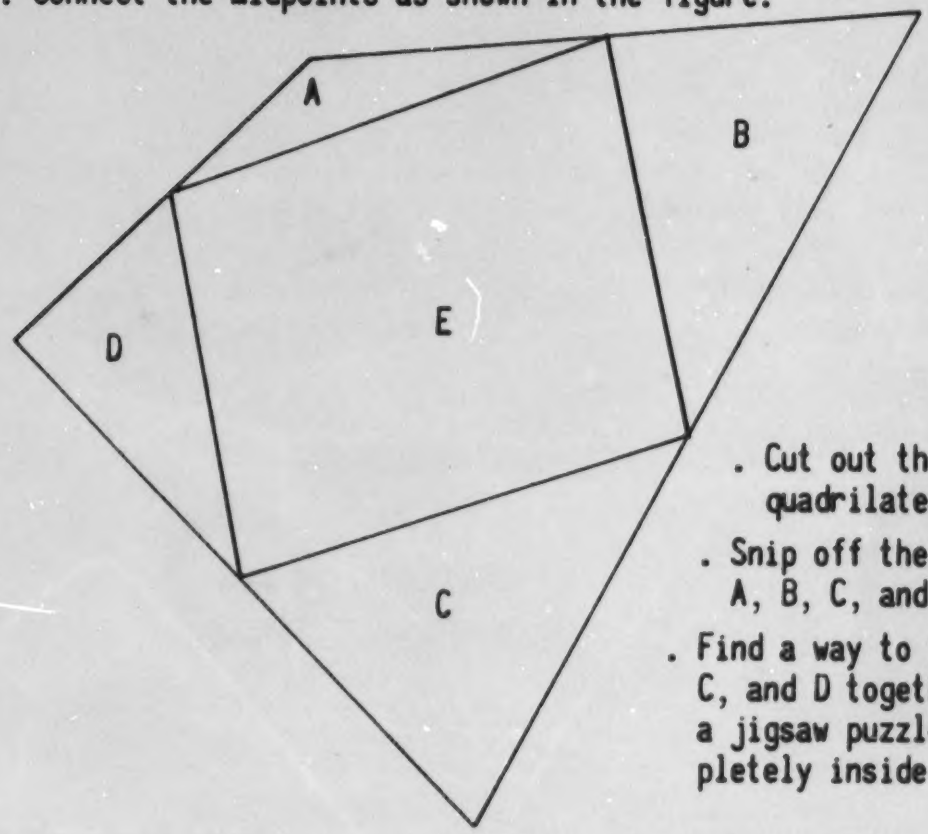
SNIPPING QUADRILATERALS

Needed: Centimetre ruler
Scissors

Follow these directions:

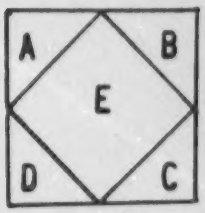
. Draw a large quadrilateral with all sides of different lengths.
. Find the middle point of each side.
. Connect the midpoints as shown in the figure.



. Cut out the large quadrilateral.
. Snip off the regions A, B, C, and D.
. Find a way to fit A, B, C, and D together (like a jigsaw puzzle) completely inside region E.

1. Write exact directions as to how you were able to fit the pieces together. (Hint: Where should piece A be placed inside region E? piece B? piece C? piece D?)
2. Experiment with another case. Do your directions still work? If not, change them so they do.

EXTENSION Sammy started with a square. He discovered that it was possible to fold regions A, B, C, and D over so as to completely fill region E. This folding method will also work for other quadrilaterals besides squares. Find as many of these as you can.



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-177-

Snipping Quadrilaterals

Mathematics teaching objectives:

- Find midpoints of line segments.
- Use slides and rotations to cover a region.

Problem-solving skills pupils might use:

- Make and use a model.
- Make generalizations based upon data.

Materials needed:

- Centimetre ruler
- Scissors

Comments and suggestions:

- Pupils may ask if it is alright to turn the pieces over. This is allowed, although no solution is possible by using this kind of transformation. They should soon discover this.
- Encourage pupils to begin with large quadrilaterals. The pieces fit together better and are easier to handle if the pieces are larger.

Answers:

1. One method: Slide pieces A and C (or pieces B and D) to opposite sides of region E. Rotate the remaining two pieces 180° and fit them along the same edge from which they were cut.
2. Experiment done by pupils.

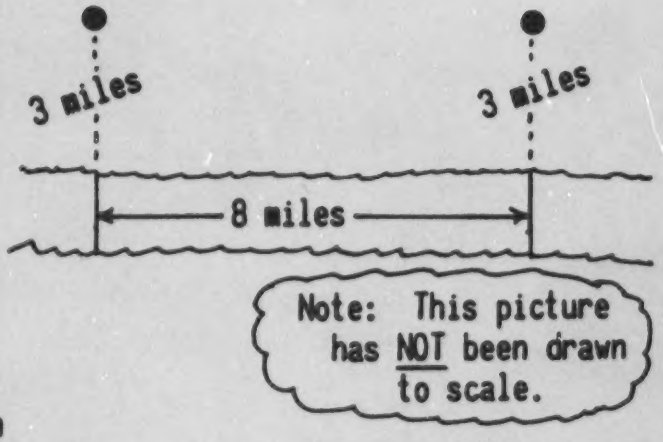
Extension:

The method will work for any quadrilateral with perpendicular diagonals.

OIL WELLS

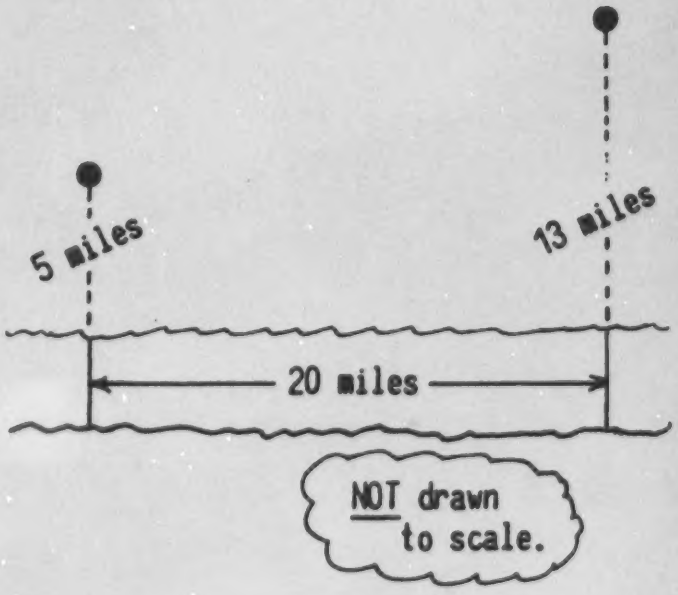
Needed: Centimetre ruler

Two oil wells are located 3 miles from the shore of a river. A single storage tank is to be built on the shore. Oil is to be piped from each well directly to the storage tank. Of course, the company wants to use as little pipe as possible.



1. What is the best location for the storage tank?
2. How much pipe is needed?

On another stretch of the river, two more oil wells have been drilled (see drawing). Another storage tank is to be built to service these two wells.



3. What is the best location for the storage tank?
4. How much pipe is needed?

Oil Wells

Mathematics teaching objectives:

- Measure line segments in millimetres.
- Use the Pythagorean Property (optional).
- Make a scale drawing.

Problem-solving skills pupils might use:

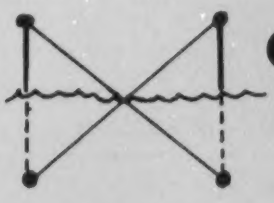
- Make necessary measurements for obtaining a solution.
- Make reasonable estimates.
- Guess and check.

Materials needed:

- Centimetre ruler
- Calculator (optional)

Comments and suggestions:

- Pupils need to construct accurate scale drawings of each situation. A suggested scale is 1 cm: 1 mile.
- Guessing is an appropriate way to determine each answer. Pupils will decide that the middle is best for problem 1. A location closer to the 5-mile well is best for problem 3. Each of these locations can be found by reflecting each oil well to the other side of the river and then connecting the wells diagonally.

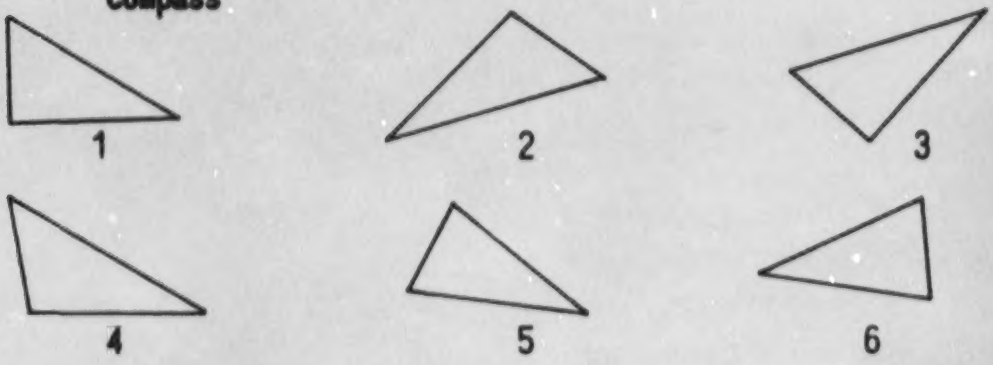


Answers:

1. 4 miles from the left-end.
2. 10 miles of pipeline.
3. Approximately 5.6 miles from the left-end.
4. Approximately 27 miles of pipeline.

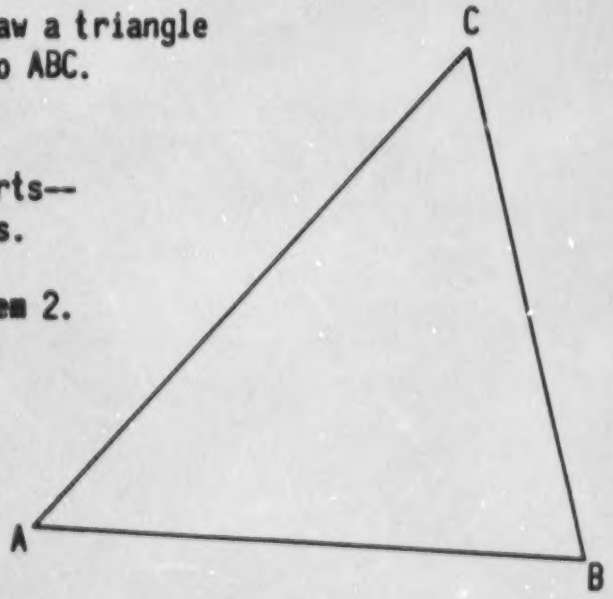
CONGRUENT TRIANGLES

Needed: A centimetre ruler
Protractor
Compass



1. Triangles are <u>congruent</u> if they have exactly the same size and shape. Which triangles above appear to be congruent?

2. Without tracing, draw a triangle that is congruent to ABC.



3. A triangle has 6 parts--
3 angles and 3 sides.

Refer back to problem 2.

a. Did you have to measure all 6 parts in order to draw your new triangle?

b. Explain the process you used and the measurements you took.

Congruent Triangles

Mathematics teaching objectives:

- Identify congruent triangles.
- Construct congruent triangles.
- Decide on necessary conditions for triangles to be congruent.

Problem-solving skills pupils might use:

- Make and use a drawing.
- Look for properties.

Materials needed:

- Centimetre ruler
- Protractor
- Compass

Comments and suggestions:

- Prerequisite skills - ability to use a protractor and compass
- Pupils should discover the traditional side-side-side, side-angle-side, and angle-side-angle congruence properties. These properties should be summarized after the pupils have completed problem 4.

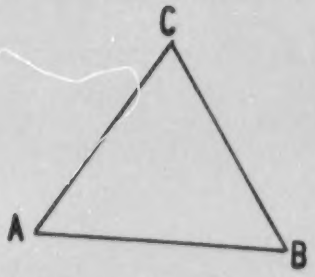
Answers:

1. Triangles 2 and 4, triangles 1 and 3, triangles 5 and 6 are congruent.

4. a. No (There is no angle-angle-angle congruency property.)
   b. Yes (side-side-side)
   c. Yes (side-angle-side)
   d. Yes (angle-side-angle)

Extension:

Pupils may use ruler and protractor.
Another method is to subdivide the figure into three triangles and then use a ruler and a compass to make the construction (side-side-side).



4. Each of these exercises refers to a triangle that is lettered like this one.

   a. Draw a triangle with
   $\angle A = 30^0, \angle B = 50^0, \angle C = 100^0$

   Compare your triangle with those made by one or two classmates. How are they alike? How different?

   b. Draw a triangle with
   AB = 4 cm, AC = 7 cm, BC = 6 cm

   Compare your triangle with those made by one or two classmates. How are they alike? How different?

   c. Draw a triangle with
   $\angle A = 60^0$, AB = 4 cm, AC = 7 cm

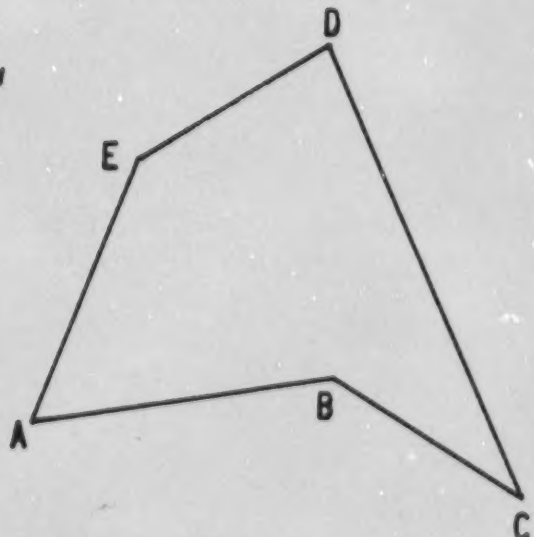
   Compare your triangle with those made by one or two classmates. How are they alike? How different?

   d. Draw a triangle with
   AB = 8 cm, $\angle A = 40^0$, $\angle B = 60^0$

   Compare your triangle with those made by one or two classmates. How are they alike? How different?

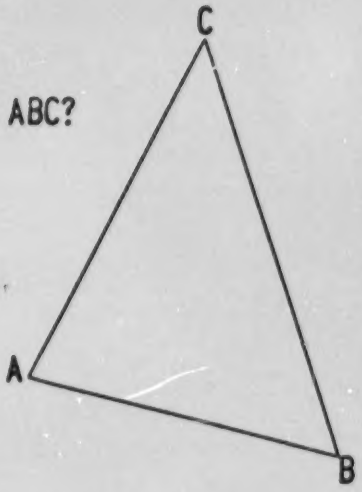
EXTENSION

Without tracing, draw a figure that is congruent to ABCDE.



DRAWING TRIANGLES

Needed: Compass and centimetre ruler



1. What is the length of each side of triangle ABC?

2. Suppose a triangle has sides of 5 cm, 6 cm, and 8 cm. Use a ruler to draw a triangle with these dimensions.

3. It is easier to use a compass to help you draw a 5-6-8 triangle. Show how this can be done.

4. If possible, draw triangles which have sides of

a. 5-5-5	d. 6-8-10	g. 8-8-3	i. 3-4-5
b. 7-7-4	e. 5-12-13	h. 3-4-7	j. 4-5-6
c. 7-4-4	f. 3-3-8		

5. Refer back to problem 4. You probably discovered that two of the triangles were impossible to draw.
   a. List the side lengths for the two impossible ones.
   b. Why was it impossible to use these lengths?
   c. Give another set of side lengths which will not form a triangle.

6. An <u>isosceles triangle</u> has at least two sides equal. Which lengths in problem 4 are for isosceles triangles?

7. An <u>equilateral triangle</u> has all sides equal. Which lengths in problem 4 are for an equilateral triangle?

Drawing Triangles

Mathematics teaching objectives:

- Construct triangles given three sides.
- Identify when three sides will not make a triangle.
- Use "isosceles," "equilateral," "scalene," and "right" to identify triangles.

Problem-solving skills pupils might use:

- Make and use a drawing.
- Classify objects.
- Make generalizations based upon data.

Materials needed:

- Centimetre ruler
- Compass

Comments and suggestions:

- Pupils may need help with the method used in problem 3. In problem 3, first draw segment AB. Then an arc equal to $\overline{AC}$ is made from A. Also, an arc equal to $\overline{BC}$ is made from B. The intersection of the two arcs locates C.

Answers:

1. 5 cm, 6 cm, 7 cm
2. Pupils' drawings
3. See the comments above.
4. All are possible except f and h.
5. a. 3-3-8; 3-4-7
   b. Answers may vary. In general, the sum of any two sides of a triangle must be greater than the third side.
   c. Answers will vary. One example is 5-6-14.
6. a, b, c, g
7. a
8. d, e, i, j
9. d, e, i
10. d, e, i, j

Extension: First subdivide the pentagon into 3 triangles by drawing the diagonals from one vertex. Then use the method described above for constructing triangles.

8. In a scalene triangle no pairs of sides are equal. Which lengths in problem 4 are for scalene triangles?

9. A right triangle has one square corner. Which lengths in problem 4 are for right triangles?

10. Which triangles suggested in problem 4 are scalene and also right triangles?

11. Draw an isosceles right triangle. Measure the lengths of the sides.

EXTENSION

Show how to use only a compass and ruler to draw a duplicate of this pentagon.

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PYTHAGORAS

Needed: Centimetre ruler

Pythagoras was a Greek mathematician who lived about 2500 years ago. He made many mathematical discoveries. One of his most famous discoveries was about certain kinds of triangles.

Study the triangles on the next page. Triangles A, B, E, F, G, and I are particular kinds of triangles. Draw another triangle that also fits this category.

Place a check in front of the 6 letters in the table that represent right triangles.

Triangle	a	b	c	a^2	b^2	c^2
A						
B						
C						
D						
E						
F						
G						
H						
I						

1. Measure the sides of each triangle to the nearest half-centimetre. Place your measurements in the left part of the table.
2. Make the computations necessary to complete the right side of the table.
3. Study the results in the right side of the table. Look closely at the numbers that correspond with the right triangle. Write about any observations you've made.
4. Do you think a 9-10-11 triangle is a right triangle? First make a prediction. Then check your prediction by making a drawing.
5. Is a 9-12-15 triangle a right triangle? First make a prediction. Then check your prediction by making a drawing.

Pythagoras

Mathematics teaching objectives:

- Identify right triangles.
- Discover the Pythagorean Property for right triangles.

Problem-solving skills pupils might use:

- Make necessary measurements for obtaining a solution.
- Use a table.
- Look for patterns.
- Make generalizations based upon data.

Materials needed:

- Centimetre ruler

Comments and suggestions:

- This lesson should be used as a class activity. Many students will need some teacher direction.
- All measurements in the triangles, except E and I, are whole numbers When measured to the nearest half-centimetre.
- In problem 3, some pupils may need a hint for finding a pattern in the table. Suggest they add the numbers in the a^2 and b^2 column for each triangle.
- An effective overhead visual can be used for presenting an informal proof of the Pythagorean relation. See page 192.

Answers:

1. & 2.

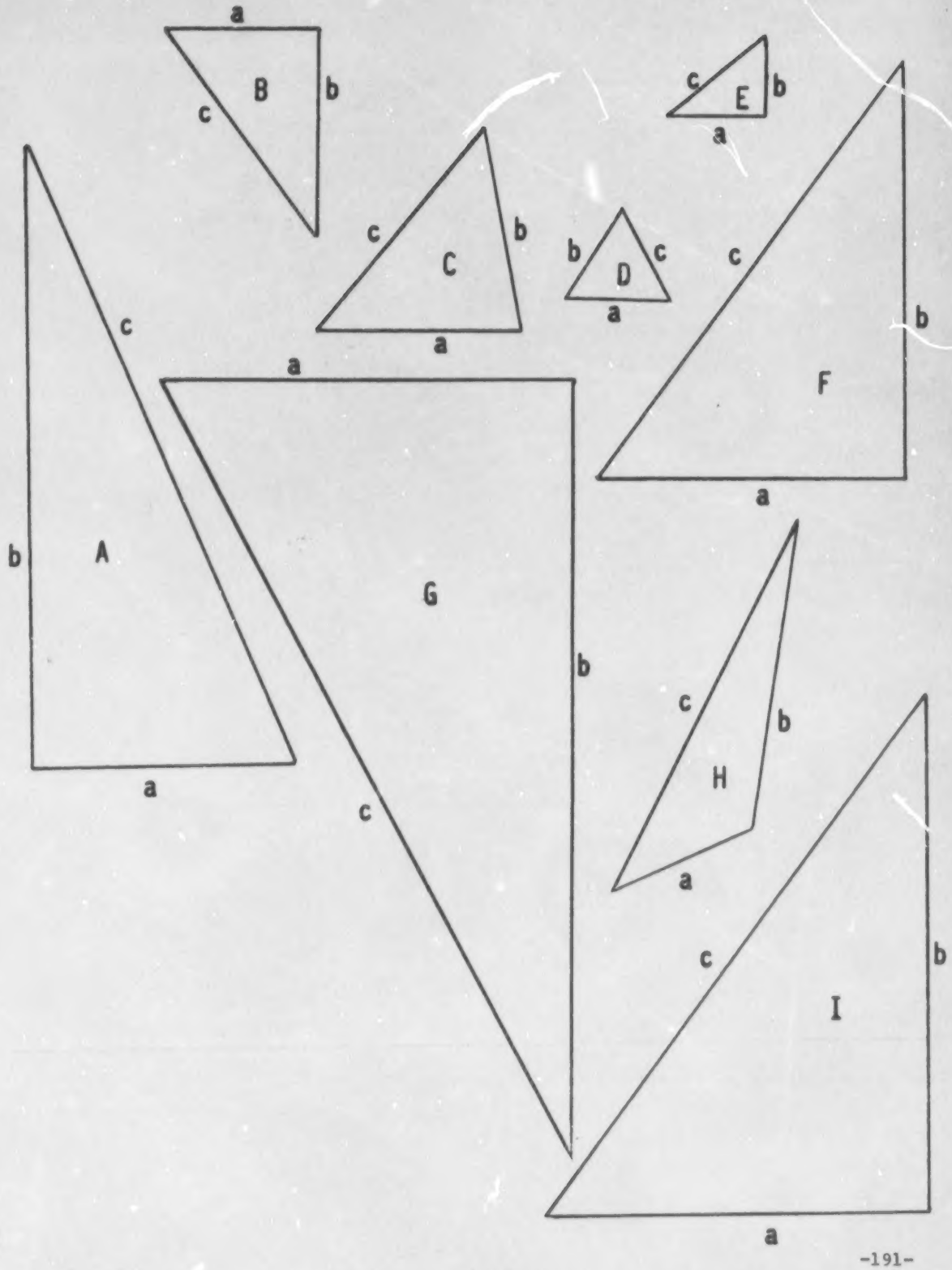
Triangle	a	b	c	a^2	b^2	c^2
✓ A	5	12	13	25	144	169
✓ B	3	4	5	9	16	25
C	4	4	5	16	16	25
D	2	2	2	4	4	4
✓ E	2	1.5	2.5	4	2.25	6.25
✓ F	6	8	10	36	64	100
✓ G	8	15	17	64	225	289
H	3	6	8	9	36	64
✓ I	7.5	10	12.5	56.25	100	156.25

3. $a^2 + b^2 = c^2$

4. No, because $81 + 100 \neq 121$.

5. Yes, because $81 + 144 = 225$.

Pythagoras (cont.)



-191-

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Pythagoras

The sequence of overhead transparencies needed for the informal proof of the Pythagorean Property follows:

Fig. a

Fig. b

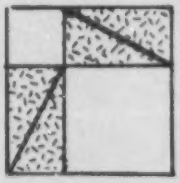


Fig. c

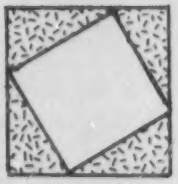
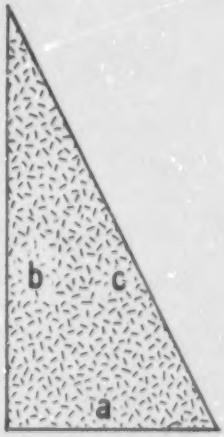
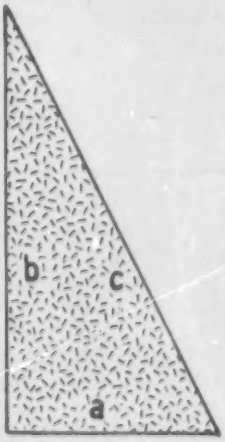


Fig. d

The areas of the light spaces in Figures b, c, and d are equal. In Figure c it is visually obvious that the "light space" area is $a^2 + b^2$, in Figure d, c^2. The formula for the light space in Figure b is not visually obvious but, of course, it could be expressed as $a^2 + b^2$ or as c^2.

Reproduce first, then cut here to use lower portion for overhead transparencies.





Grade 8

VIII. CALCULATOR

VIII. CALCULATOR

Calculators are here to stay!! The ever decreasing cost of four function calculators make them more and more attractive as a tool to use in the mathematics classroom. Many investigations or activities that once would take a lot of time are now feasible. Problem solving can be emphasized because the tediousness of computation is removed from the problem by using the calculator.

The six activities in this section use whole numbers and decimals. A knowledge of how to use a calculator is assumed. If your pupils have not used them before, several readiness activities with calculators would be appropriate. Several calculator books now on the market can provide readiness activities. One, in particular, is CALCULATOR ACTIVITIES FOR THE CLASSROOM by Immerzeel and Ockenga, available from Creative Publications.

Special emphasis should be placed on making estimates before and after using the calculator. A way to motivate estimation is to ask pupils how they know a calculator gives the right answer. Discussion can bring out that people can push the wrong buttons and use the calculator incorrectly. Weak batteries also can cause incorrect answers. Estimating answers can help in catching errors.

Pupils may need to review some techniques for estimating. Perhaps the best way to accomplish this is for them to observe the techniques you use in estimating--like rounding up or down, or changing a problem into one that is easy to solve mentally.

Several of these activities can be done without calculators, if your pupils have good basic skills with whole numbers and decimals. They will have to think about each problem more carefully to avoid a lot of tedious computation.

The activities can be done in any order and at most anytime during the year. However, we do suggest that consideration be given to using them in the indicated order.

A NUMBER TIMES ITSELF

Felix is known as the class "trickster." Here are some problems he gave the class.

- What number times itself is equal to 81?
- What number times itself is equal to 25?
- What number times itself is equal to 16?
- What number times itself is equal to 20?

1	x	1	=	1
2	x	2	=	4
3	x	3	=	9
4	x	4	=	16
5	x	5	=	25
6	x	6	=	36
7	x	7	=	49
8	x	8	=	64
9	x	9	=	81
10	x	10	=	100

"Impossible, ... absurd ... ridiculous ... can't be done." These were some of the responses to Felix's last problem.

"Aha," replied the Trickster, "Haven't you ever heard of decimals?"

1. In Felix's last problem, which of the following is the best answer?

   4.6 x 4.6 4.5 x 4.5 4.4 x 4.4

2. Use decimals (tenths) to find the "best answer" to each of these. Remember, in each case, the number in the two boxes must be the same.

   a. ☐ x ☐ = 30 c. ☐ x ☐ = 50

   b. ☐ x ☐ = 40 d. ☐ x ☐ = 60

3. Use decimals (hundredths) to find the "best answer" to each part in exercise 2.

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A Number Times Itself

Mathematics teaching objectives:

- Use a calculator.
- Estimate using decimals.

Problem-solving skills pupils might use:

- Make reasonable estimates.
- Guess and check.
- Use a table.

Materials needed:

- Calculator

Comments and suggestions:

- Do not suggest the use of the square root button. If it is used, this activity does not accomplish its intended purpose.
- Encourage use of the table. Pupils may see in 2-a, for example, that since 30 is about halfway between 25 and 36, the decimal wanted should be about halfway between 5 and 6.
- Two problem-solving strategies will be used. Some pupils will guess, use the calculator to check, and then refine the guess. Others will use the data in the table, recognize limits, and apply what they know about decimals to arrive at the answer.

Answers:

1. 4.5 x 4.5

2. a. 5.5 b. 6.3 c. 7.1 d. 7.7

3. a. 5.48 b. 6.32 c. 7.07 d. 7.75

BEST ESTIMATE

Play this game in small groups.

Each player estimates the answer to Problem 1. Be sure to write down your estimates. You should decide how much time to allow for each estimate.

Use a calculator to determine the correct answer.

The player whose estimate is closest gets one point.

Continue with the rest of the problems using the same rules.

1. $378 + 456 + 924 + 492$
2. $34 \overline{)\$182.92}$
3. 3^6
4. $6{,}429 \div 9{,}980$
5. $70{,}009 - 4{,}280$
6. 3.9×29.8
7. $\sqrt{82}$
8. $18 \div 15$
9. 2^{15}
10. $1.25 \div 32.1 \div .992$
11. 20.1×6.24
12. $\sqrt{106}$
13. $20 \overline{)94{,}020}$
14. 5^6
15. $20.01 \times 20.01 \times 20.01$
16. $\$200.75 - \$159.88$

Best Estimate

Mathematics teaching objectives:

- Use a calculator.
- Estimate.

Problem-solving skills pupils might use:

- Make reasonable estimates.

Materials needed:

- Calculator

Comments and suggestions:

- A discussion of skills used in estimation, that is, rounding, computing with multiples of 10, etc., makes a good lead-in for this activity.
- Each pair of pupils needs a calculator. After estimates are made by both players, they use the calculator to find the exact answer. The calculator can also be used to determine how far off each estimate is.
- In keeping with the idea that no estimate is wrong, a variation of the scoring would be to award two points for the closer estimate and one point for the other estimate.
- Pupils enjoy creating problems which they think will be too difficult for others. Have them create ten or twenty problems to be used with this game.

Answers:

Pupil estimates will vary. The answers shown are exact.

1. 2250
2. $5.38
3. 729
4. 16,409
5. 65,729
6. 116.22
7. 9.0553851
8. 1.2
9. 32,768
10. 34.342
11. 125.424
12. 10.29563
13. 4701
14. 15,625
15. $012.006
16. $40.87

DISCOVERING PATTERNS

Study each pattern.
Does it continue?
Try more examples.
Check them by using a calculator.

1.
$$1 + 2 = 3$$
$$4 + 5 + 6 = 7 + 8$$
$$9 + 10 + 11 + 12 = ? + ? + ?$$
$$\vdots \qquad\qquad \vdots$$

2.
$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = ? + ? + ?$$
$$? + ? + ? + ? + ? = 41^2 + ? + ? + ?$$
$$\vdots \qquad\qquad \vdots$$

3.
$$3^2 + 4^2 = 5^2$$
$$3^3 + 4^3 + 5^3 = 6^3$$
$$\vdots \qquad \vdots$$

Discovering Patterns

Mathematics teaching objectives:

- Use a calculator.
- Discover mathematical patterns.

Problem-solving skills pupils might use:

- Look for patterns.
- Make conjectures based on data.

Materials needed:

- Calculator

Comments and suggestions:

- Pupils will need to record partial answers if the calculator being used does not have a memory.
- Each problem has several patterns. For example, consecutive numbers are used in problem 1. Each line of the third problem begins with 3 and uses as many terms on the left side as the power being used.

One pattern in problem 2 can be seen by looking vertically at the numbers to the left of the equal sign. Note that the bases differ first by 8, then by 12, by 16, by 20, etc.

$4^2 =$
$12^2 =$
$24^2 =$
$40^2 =$
$60^2 =$

Answers:

1. $1 + 2 = 3$

$4 + 5 + 6 = 7 + 8$

$9 + 10 + 11 + 12 = 13 + 14 + 15$

$16 + 17 + 18 + 19 + 20 = 21 + 22 + 23 + 24$

The pattern will continue.

2. $3^2 + 4^2 = 5^2$

$10^2 + 11^2 + 12^2 = 13^2 + 14^2$

$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$

$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$

$55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 = 61^2 + 62^2 + 63^2 + 64^2 + 65^2$

The pattern will continue.

3. $3^2 + 4^2 = 5^2$ (Yes)

$3^3 + 4^3 + 5^3 = 6^3$ (Yes)

$3^4 + 4^4 + 5^4 + 6^4 = 7^4$ (No)

The pattern does not continue.

MISSING PARTS

The numbers in each column are to be used to complete the problems involving that operation. For example, the numbers in the addition column are to be used to complete the addition problems. An extra number is contained in each column.

Addition	Subtraction	Multiplication	Division
84.59	9.9	3.7	6
73.9	8.93	6	15.6
15.7	5.93	8.08	.8
18.77	17.2	5.88	3
79.2	4.6	9.31	14.6
89.9	12.12	1.9	9.66
97.6	20	2.14	3.2

1. ◯ + ◯ = 105.6	2. ◯ − ◯ = 8.27	3. ◯ x ◯ = 7.918
4. ◯ ÷ ◯ = 5.2	5. ◯ + 23.7 = ◯	6. ◯ ÷ 1.61 = ◯
7. ◯ − 14.07 = ◯	8. ◯ x .98 = ◯	9. 14.5 − ◯ = ◯
10. 2.56 ÷ ◯ = ◯	11. 5.39 + ◯ = ◯	12. 4.9 x ◯ = ◯

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Missing Parts

Mathematics teaching objectives:

- Estimate.
- Use a calculator.
- Use inverse operations.

Problem-solving skills pupils might use:

- Make reasonable estimates as answers.
- Guess and check.
- Work backwards

Materials needed:

- Calculator

Comments and suggestions:

- Two distinct problem-solving strategies are possible. Some pupils will use a guess and check approach. Others will analyze the data, recognize limits, and apply what they know about basic facts.
- Remind pupils that each column contains an extra number. They can cross off each number as it is used.
- Pupils might be challenged to produce a set of problems similar to these to share with others.

Answers:

1. $15.7 + 89.9 = 105.6$ or $89.9 + 15.7 = 105.6$
2. $17.2 - 8.93 = 8.27$
3. $2.14 \times 3.7 = 7.918$ or $3.7 \times 2.14 = 7.918$
4. $15.6 \div 3 = 5.2$
5. $73.9 + 23.7 = 97.6$
6. $9.66 \div 1.61 = 6$
7. $20 - 14.07 = 5.93$
8. $6 \times .98 = 5.88$
9. $14.5 - 9.9 = 4.6$ or $14.5 - 4.6 = 9.9$
10. $2.56 \div 3.2 = .8$ or $2.56 \div .8 = 3.2$
11. $5.39 + 79.2 = 84.59$
12. $4.9 \times 1.9 = 9.31$

CALCULATOR PUZZLERS

1. Verify that the sum of these seven numbers is 636.

   $27 + 108 + 29 + 265 + 19 + 35 + 153$

   Now use seven different numbers. Make them all 2-digit numbers. The sum should still be 636.

2. How many times can 29 be subtracted from 1015?

3. Find all the whole number divisors of 323.

4. Find all the whole number divisors of 293.

5. Which of these answers can be displayed on your calculator?

   $(589)^2$ $(3123)^2$ $(856194)^2$

   What is the largest N such that N^2 can be displayed on your calculator?

6. What is the larges N such that N^3 can be displayed on your calculator?

7. Find two numbers a and b such that $a^b = b^a$.
   Try these.

   a. 4^8 and 8^4

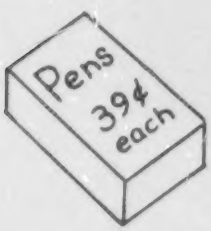
   b. 5^4 and 4^5

   c. 10^2 and 2^{10}

   d. 6^3 and 3^6

   e. Try some examples of your own.

EXTENSION



The price for a single pen is 39¢.
This price is reduced if pens are purchased in quantity.
The price per pen is always an exact number of cents.
A person buys all the pens in the box for $22.91.
How many pens were in the box?

Calculator Puzzlers

Mathematics teaching objectives:

- Use a calculator.
- Use basic computation concepts.

Problem-solving skills pupils might use:

- Look for patterns.
- Guess and check.

Materials needed:

- Calculator

Comments and suggestions:

- Pupils probably will use two approaches to these problems. Some will "grind-it-out-the-hard-way" while others will use some math concepts to simplify the work. For example, problem 2 can be worked by actually subtracting 29 as many times as possible. Or if repeated subtraction is recognized as division, pupils can simply divide 1015 by 29 and look at the whole number part of the answer.
- The answers shown below are for a calculator that displays eight digits. Adjustments are necessary if a ten- or twelve-digit display is available.

Answers:

1. Answers will vary. One possible answer is 90 + 91 + 92 + 93 + 94 + 95 + 81.
2. 35 times
3. 1, 17, 19, 323
4. 293 is prime. Its only divisors are 1 and 293.
5. $(589)^2 = 346,921$ $(3123)^2 = 9,753,129$ $(856,194)^2$ cannot be displayed. The largest whole numbers is 9999.
6. The largest whole number is 464.
7. a. $4^8 = 65536$ $8^4 = 4096$ $4^8 \neq 8^4$
   b. $5^4 = 625$ $4^5 = 1024$ $5^4 \neq 4^5$
   c. $10^2 = 100$ $2^{10} = 1024$ $10^2 \neq 2^{10}$
   d. $6^3 = 216$ $3^6 = 729$ $6^3 \neq 3^6$

Extension:

The prime factorization of 2291 is 79 x 29. Therefore, the number of pens in the box is 79.

I'M THINKING OF A NUMBER

1. If you add the number to itself, the result is 5896.
   What is the number?

2. If you multiply the number by 17 and add 93, the result is 501.
   What is the number?

3. If you add the number to itself and then multiply this result by 16, the answer is 4064.
   What is the number?

4. If you divide the number by 23 and then subtract 31, the result is 0.
   What is the number?

5. If you add twice the number to three times the number, the result is 1335.
   What is the number?

6. Twice the number added to 73 is the same as three times the number added to 35.
   What is the number?

7. If you multiply the number by itself, the result is 3481.
   What is the number?

8. If you multiply the number by itself, the result is very close to 90.
   What is the best number when expressed in
   a. tenths
   b. hundredths
   c. thousandths
   d. ten-thousandths

EXTENSION

Make up your own "I'm thinking of a number" problem. Write it down and hide the answer. Then give it to a classmate to solve.

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I'm Thinking Of A Number

Mathematics teaching objectives:

- Use a calculator.
- Use inverse operations.

Problem-solving skills pupils might use:

- Guess and check.
- Make and use a table.
- Work backwards.

Materials needed:

- Calculator

Comments and suggestions:

- If pupils use guessing and checking, encourage them to record their results in a table. Writing down the results helps in making refinements.
- Do not suggest the use of the square root button for problem 7. However, do suggest the use of the square root button for problem 8 after pupils struggle for a while.

Answers:

1. 2948
2. 24
3. 127
4. 713
5. 267
6. 48
7. 59
8. a. 9.5 c. 9.487
   b. 9.49 d. 9.4868

You may wish to mention that the answer is more exact as one continues with more decimal places.

Extension:

Answers will vary.

Grade 8

IX. PERCENT ESTIMATION

IX. PERCENT ESTIMATION

Percents are everywhere - but how many times do you need to work out exact answers? You see a sign on a department store window which says:

> After Christmas Sale -
> 25% off on stereos with tape decks.

The set you want is marked $389.89.

Do you get out a piece of paper to figure out how much to pay? Probably not. You would likely reason

25% means $\frac{1}{4}$ off.

$\frac{1}{4}$ of $400 is $100.

Therefore, the price is around $290.

The estimate of $290 is good enough to help convince you whether or not you would like to purchase the stereo.

The main purpose of the lessons in this unit is to help pupils make estimations when percents are involved. We recommend that this section be done before a formal treatment is given where "exact" answers are the goal of instruction. A formal approach, if taught first, seems to work against the flexibility of thought needed in estimating.

Good percent estimators do not confine themselves to one set method for making approximations. They use different methods, depending upon the context of the problem and their personal preferences. Therefore, good practice for "would-be" estimators is to have them look at percent from different points of view. In the first three lessons three different estimation methods are presented. In the last two lessons pupils are encouraged to use whatever method seems natural to them. On the following page three different methods are illustrated with three different problems.

The following problem is solved using the three different methods presented in this unit.

Bob made a 10% commission on a Buick he sold for $1595.
Estimate his commission.

- Meaning of Percent Method (or "for every hundred" method)

  1595 is about 1600.

  10% means 10 for every hundred.

  Since there are 16 hundreds, the commission is 16 x 10 or $160.

- One-Percent Method

  100% is approximately 1600.

  1% is 16.

  10% is 10 x 16 or $160.

- Fraction method

  10% is $\frac{10}{100}$ or $\frac{1}{10}$.

  $\frac{1}{10}$ of 1600 is $160.

As you study the different ways for solving the problem above, you probably will favor one method over the others. Also, it is unlikely that the preferred method is the same for all percent problems. This too will be true of your pupils.

Before presenting this section to your class we suggest that you read the Percent Sense unit from the 7th-grade PSM packet. If your pupils are unfamiliar with this unit, you may wish to use some of the lessons prior to starting Percent Estimations.

The prerequisites for the lessons which follow are the ability to multiply and divide by 100 and to demonstrate a knowledge of the basic meaning of simple common fractions. Also, some estimation exercises should be given prior to using this unit. The lessons called "Best Guess" will provide some good estimation experience. These are found in the 8th-grade Drill And Practice unit (pp 59-77).

We suggest that the lessons be done in the order indicated. The answers in the page commentaries are given in two forms. First, a possible range is given for what we think are reasonably good estimates. However, the range should not be treated as a strict guide for determining whether the estimates are right or wrong. Remember, estimates are not wrong--some are just better than others. We also have given the "exact" answers after the usual rules of rounding have been applied.

MEANING OF PERCENT

Estimation is one of the most practical applications of mathematics. In this unit, you will learn several estimation methods useful in situations dealing with percent.

Remember--don't be afraid to make estimates. The estimate you make is not wrong. It may not be as close as you'd like, but you'll get better with more practice.

Miss Green often eats out on Sunday afternoon. Last Sunday her bill was $7.95. Miss Green usually leaves a tip of about 10%. Which amount below do you think is closest to 10% of the bill?

Why is an $8.00 tip unreasonable?

Why is a $4.00 tip unreasonable?

If you thought an $.80 tip is most appropriate, you were correct. Perhaps you arrived at your estimate in the following way:

$ 7.95 is close to $ 8.00.
10% means 10¢ for every dollar
or $.80 for every $ 8.00.

This estimation method is called the MEANING OF PERCENT method.

Meaning of Percent

Mathematics teaching objectives:

- Use rounding and mental arithmetic in making estimates.
- Estimate answers to percent problems in a variety of settings.
- Use the "meaning of percent" method for making estimates.

Problem-solving skills pupils might use:

- Make reasonable estimates.
- Solve an easier but related problem.

Materials needed:

- None

Comments and suggestions:

- Help pupils overcome some of their reluctance to make estimates by reminding them that often problems can be made easier if the numbers are rounded. One way to teach pupils to round and estimate is by making it obvious to them that you get reasonable answers that way. This lesson should help accomplish this objective.
- The "meaning of percent" method is a natural way to think of percent. "Per" in percent is then used in the same way as the "per" in per hour or per gallon.
- After working the Class Exercises, pupils should be able to do the remaining exercises individually. After pupils have done what they can on their own, they might profit from comparing their answers with one or more of their classmates.
- All of the word problems are examples of the type, "Find a given percent of a given number."

Answers:

The "exact" answers are given in parentheses.

1. a. $.55 to $.65 ($.60)
   b. $1.20 to $1.30 ($1.23)
   c. $.30 to $.35 ($.31)
   d. $1.40 to $1.50 ($1.45)

2. a. $.20 to $.25 ($.21)
   b. $.30 to $.35 ($.34)
   c. $.35 to $.40 ($.39)
   d. $.55 to $.60 ($.56)

Class Exercises

1. Remember, Miss Green usually leaves a tip of about 10%. Complete the chart below. Use the Meaning of Percent method.

Cost of Meal	Approximate Tip
a. $ 5.95	?
b. $12.25	?
c. $ 3.10	?
d. $14.50	?

2. Mr. Miser usually leaves a tip of about 5%. Complete this chart. Use the Meaning of Percent method.

Cost of Meal	Approximate Tip
a. $ 4.15	?
b. $ 6.75	?
c. $ 7.85	?
d. $11.25	?

Exercises

In each exercise, use the Meaning of Percent method to help you find an approximate answer.

1. A 12% downpayment is required on all items. Remember,
   12% means 12¢ on every dollar, or
   $12 on every $100.

   Complete this chart.

Item	Cost	Approximate Downpayment
T.V.	$ 195	
Stereo	$ 415	
Davenport	$ 799	
Water Bed	$ 550	

Answers: (cont.)

Exercises

1.

TV	$20 to $24	($23.40)
Stereo	$48 to $50	($48.90)
Davenport	$95 to $96	($95.88)
Water Bed	$60 to $72	($66.00)

2.

Cost	Down Payment	
$ 312	$ 36 to $40	($37.44)
$ 489	$ 45 to $50	($48.90)
$ 79.79	$ 11 to $13	($11.97)
$ 49.95	$ 9 to $11	($9.99)
$ 2390	$110 to $130	($119.50)
$ 305	$150 to $160	($152.50)
$ 299.99	$100 to $110	($105.00)

3.

Music	260 to 320	(291)
Science	220 to 250	(236)
Shop	150 to 170	(165)
Mathematics	350 to 370	(354)
Art	110 to 130	(118)
Home Economics	70 to 80	(75)

4. a. $ .50 to $ .60 ($ .55) b. $ .60 to $ .80 ($ .69)
c. $1.20 to $1.30 ($1.26) d. $1.10 to $1.20 ($1.13)

5.

Ford	$ 75 to $85	($79.60)
Buick	$100 to $110	($103.60)
VW	$ 50 to $60	($55.60)
Honda	$150 to $170	($159.60)

6. Answers will vary.

2. Different stores require different downpayments. Estimate the downpayment on each item whose cost is given in the chart. Remember, use the "Meaning of Percent" method.

Cost	% Down	Approximate Downpayment
$ 312	12%	
$ 489	10%	
$ 79.79	15%	
$ 49.95	20%	
$2390	5%	
$ 305	50%	
$ 299.99	35%	

3. The enrollment of Adams School is 393. The principal reported the percents of students enrolled in certain subjects. In each case, determine the approximate number of students enrolled.

	Percent of Student Body Enrolled	Approximate Number of Students
Music	74%	
Science	60%	
Shop	42%	
Mathematics	90%	
Art	30%	
Home Economics	19%	

4. Mrs. Young frequently uses a taxi for her transportation around town. She normally tips about 20%. What would be a reasonable tip for each of these fares?
   a. $2.75 b. $3.45 c. $6.30 d. $5.65

5. Bob makes an 8% commission on all used cars that he sells. This means that he gets to keep $8 out of every $100. Estimate the commission Bob makes on the sales.

Car	Price	(Estimate) Commission
Ford	$ 995	
Buick	$1295	
VW	$ 695	
Honda	$1995	

6. Make up a percent problem and estimate the answer by using the "meaning of percent" method.

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ONE-PERCENT METHOD

Miss Haines needs to borrow $790 and the Easy-Cash Loan Company is happy to help out. Their interest charge is only 3% per month.

If Miss Haines paid back the $790 within a month, about how much interest would she have to pay?

$15.00 $20.00 $24.00 $35.00 $400.00

Here's one way to make a reasonable estimate.

$790 is about $800.
1% of $800 is $ 8.
3% of $800 is $24.

This estimation method is called THE ONE-PERCENT METHOD.

Class Exercises

1. Another loan company advertises loans at $2\frac{1}{2}\%$ per month.
   a. Use the 1% method to estimate the interest on a $790 loan.
   b. About how much interest would you have to pay on a $1210 loan?

2. Mr. Gomez pays professional dues of $\frac{1}{2}\%$ of his annual salary. About how much does he pay in dues if his annual salary is $19,100? (Use the 1% method.)

3. Mrs. Jones is playing some "Guess My Number" games with her class. See how close you can guess the numbers Mrs. Jones is thinking of.
   a. 1% of it is 5. (How large is her number? In other words,
   b. 1% of it is 7. how much is 100% if 1% of it is 5?)
   c. 1% of it is 29.
   d. 1% of it is 73.

4. See how close you can "Guess My Number."
   a. 2% of it is 8. (First find 1%; then 100%.)
   b. 3% of it is 6. c. 5% of it is 35. d. 8% of it is 24.

One-Percent Method

Mathematics teaching objectives:

- Determine approximate answers to a variety of percent word problems.
- Use rounding and mental arithmetic in making estimates.
- Use fractional percents in solving word problems.
- Use the "one-percent" method for making estimates.

Problem-solving skills pupils might use:

- Make reasonable estimates.
- Solve an easier but related problem.

Materials needed:

- None

Comments and suggestions:

- The method of finding fractional percents is easier for most pupils if the process used does not require the conversion of the percent to a decimal. The one-percent method does not require conversion to a decimal but does require the ability to divide by 100.
- Some of the word problems require pupils to find the 100% number when a percent of the number is given. (See problems 3 and 4 in the Class Exercises and problem 5 in the Exercises.) This difficult percent problem type usually can be solved rather easily by using the one-percent method.
- Pupils should be able to proceed independently on the Exercises after being introduced to the One-Percent Method using a group discussion approach to the introduction and the Class Exercises.

Answers:

Class Exercises

1. a. \$19 to \$20 (\$19.75) b. \$30 to \$31 (\$30.25)

2. \$90 to \$100 (\$95.50)

3. a. 500 b. 700 c. 2900 d. 7300

4. a. 400 b. 200 c. 700 d. 300

Exercises

Use the 1% method.

1. Mr. McKay can save 2% of his fuel bill if he pays within 10 days. About how much would he save on a fuel bill of $119.75?

2. Some banks pay different interest rates on savings accounts. The chart below shows savings account information of several students. In each case, find the approximate amount of interest.

	Savings	Percent Interest Per Year	Approximate Interest Per Year
Linda	$ 293	7%	
Bernie	$ 105	6%	
Wayne	$ 303	9%	
Lois	$ 121	5%	
Pat	$ 148	8%	

3. About how many pounds of butterfat are there in 608 pounds of milk if the butterfat content is $1\frac{1}{2}$%?

4. Mr. Cornelius is paid a base salary of $800 per month. He also receives a commission of 3% on all sales. Estimate his commission for each of the months listed below. Remember, use the 1% method.

	Sales	Commission
January	$ 3976	
February	$ 4899	
March	$ 3015	
April	$ 5542	

5. Bobbie is thinking of some numbers. See how close you can guess the numbers she's thinking of.

   a. 2% of it is 6.
   b. 6% of it is 24.
   c. 8% of it is 16.
   d. 7% of it is 35.

6. Make up a percent problem and estimate the answer by using the 1% method.

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Answers: (cont.)

Exercises

1. $2.00 to $2.50 ($2.40)

2. Linda $20 to $21 ($20.51)
   Bernie $ 6 to $ 7 ($ 6.30)
   Wayne $27 to $28 ($27.27)
   Lois $ 6 to $ 7 ($ 6.05)
   Pat $11 to $12 ($11.84)

3. 8 to 10 lbs (9.12 lbs)

4. January $110 to $130 ($119.28)
   February $140 to $160 ($146.97)
   March $ 80 to $100 ($ 90.45)
   April $150 to $180 ($166.26)

5. a. 300 b. 400 c. 200 d. 500

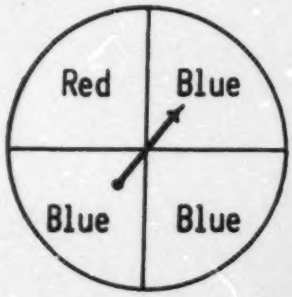
6. Answers will vary.

PERCENTS AND FRACTIONS

Mr. Bransom gave each of four students a spinner exactly like the one pictured. He asked each of them to spin it a total of 100 times. Here are the results:

Lonnie - Red 23 times
Mike - Red 53 times
Mark - Red 49 times
Tammy - Red 26 times

Do you think that these results are reasonable?



Spinner A

Is it possible that Mr. Bransom might have "fixed" a couple of the spinners? If so, which ones?

If the spinners had not been fixed, then

- red should have occurred about $\frac{1}{4}$ of the time.
- $\frac{1}{4}$ of the time means about 25 times out of 100 (25%).

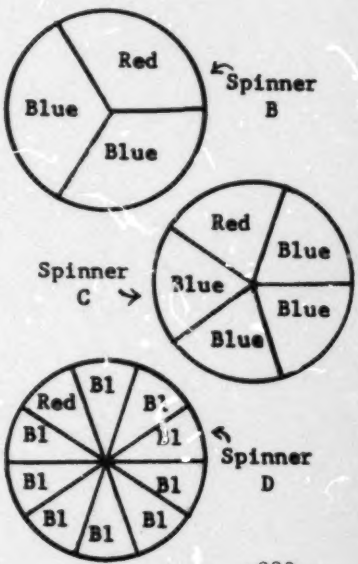
Fractions are very useful in making estimates in problems involving percent. We will call this estimation method the FRACTION METHOD.

Class Exercises

1. If the spinners were not fixed,
   a. about what fraction of the time should a blue occur?
   b. about what percent of the time should a blue occur?

2. Assume that the spinners at the right are not fixed. Complete the chart below.

Spinner	Fraction (Red)	% (Red)	Fraction (Blue)	% (Blue)
A	$\frac{1}{4}$	25%	$\frac{3}{4}$	75%
B				
C				
D				



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Percents And Fractions

Mathematics teaching objectives:

- Determine approximate answers to a variety of percent word problems.
- Use rounding and mental arithmetic in making estimates.
- Use fraction equivalents in solving percent problems.

Problem solving skills pupils might use:

- Make reasonable estimates.
- Solve an easier but related problem.

Materials needed:

- None

Comments and suggestions:

- Estimates for answers to certain percent problems can be made quickly if the fractional equivalents for certain percents are known. Especially useful are the percent equivalents for halves, thirds, fourths, fifths, and tenths. Pupils also will need to make mental calculations such as $\frac{1}{3}$ of 90 or $\frac{3}{4}$ of 60.

- Several of the word problems require the pupils to find the percent when the "part and whole" are given. This type was not included in the two previous lessons.

- Spend some classtime with the percent problems pupils make up. Often some of them show much imagination!

Answers:

Class Exercises

1. a. $\frac{3}{4}$ b. 75%

2. B. $\frac{1}{3}$, 33%, $\frac{2}{3}$, 67%

   C. $\frac{1}{5}$, 20%, $\frac{4}{5}$, 80%

   D. $\frac{1}{10}$, 10%, $\frac{9}{10}$, 90%

3. Many students at Lincoln School were absent during a flu epidemic. The total enrollment of Lincoln is 594. For each day, determine about how many students were absent.

Day	% Absent	Number of Absences (Estimate)
Monday	50%	
Tuesday	33%	
Wednesday	25%	
Thursday	20%	
Friday	10%	

4. Debbie tries to save about 25% of her earnings. About how much does she save each month?

Month	Earnings	Amount Saved (Estimate)
June	$79	
July	$61	
August	$102	

5. Coach Smith's basketball players will get an award if they make 40% of their shots. Who deserves an award?

Player	Shots Made	Shots Attempted
Julie	16	32
Wendy	10	40
Alison	9	27

Exercises

Use the Fraction Method to help you make estimates.

1. Miss Jacobs usually leaves a tip of about 10%. Complete the chart.

Cost of Meal	Approximate Tip
$ 7.25	
$ 13.85	
$ 3.20	

Answers:

Class Exercises (cont.)

3.

Monday	295 to 300	(297)
Tuesday	190 to 200	(196)
Wednesday	140 to 150	(149)
Thursday	110 to 125	(119)
Friday	50 to 60	(59)

4

June	$19 to $20	($19.75)
July	$15 to $16	($15.25)
August	$25 to $26	($25.50)

Exercises

1.

$ .70 to $ .80	($ .73)
$1.30 to $1.40	($1.39)
$ .30 to $ .35	($ .32)

2.

$3.75 to $4.25	($3.99)
$3.00 to $3.25	($3.04)
$5.00 to $5.25	($5.11)

3.

$ 79 ⟶ 24 to 26%	(25%)
$120 ⟶ 50 to 52%	(51%)
$ 41 ⟶ 9 to 11%	(10%)
$ 51 ⟶ 15 to 20%	(20%)

4.

$175.00 ⟶ $55 to $60	($58.33)
$ 89.50 ⟶ $29 to $31	($29.83)
$ 5.99 ⟶ $2	($2.00)

5.

$ 79.98 ⟶ $20	($20.00)
$ 57.98 ⟶ $14 to $15	($14.50)
$ 25.00 ⟶ $ 6 to $ 7	($ 6.25)
$199.98 ⟶ $49 to $50	($50.00)

6. Less than 50%

7. $800 ($793.33)

8.

Jay	60 to 70%	(66%)
Joan	55 to 65%	(58%)
Jill	70 to 80%	(75%)
Julie	45 to 55%	(49%)
Jerry	80 to 90%	(84%)
Jim	30 to 40%	(32%)

Percents And Fractions (cont.)

2. Big Spender usually leaves a tip of about 25%. Complete the chart.

Cost of Meal	Approximate Tip
$ 15.95	
$ 12.15	
$ 20.45	

3. Joan kept a record of her earnings and how much she was able to save. In each case, make an estimate as to the percent saved.

Earnings	Saved	Approximate Percent
$ 29	$ 10	33%
$ 79	$ 20	
$120	$ 61	
$ 41	$ 4	
$ 51	$ 10	
$ 59	$ 20	

$29 is about $30. 10 out of 30 can be expressed as $\frac{1}{3}$, or about 33%.

4. A down payment of $33\frac{1}{3}\%$ is required on all items. Complete the chart.

Cost of Item	Approximate Down Payment
$ 175.00	
$ 89.50	
$ 5.99	

5. Everything is 25% off. Determine the approximate savings on each item.

Cost	Approximate Savings
$ 79.98	
$ 57.98	
$ 25.00	
$199.98	

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6. George usually makes about 50% of his free throws. One game he made 5 out of 11. Was this more or less than his 50% average?

7. Miss Collins sold her car for $ 595. This is about 75% of what she paid for it. About how much did she pay for the car?

8. Six archers are going to take 100 shots. An award will be given to the person having the most bull's-eyes. Their records up to this point are given in the chart. In each case, estimate the percent. Then decide who you think will win the award.

Name	Bull's-eyes	Number of Shots	Approximate Percent
Jay	40	61	
Joan	30	52	
Jill	61	81	
Julie	35	71	
Jerry	41	49	
Jim	9	26	

9. Make up a percent problem and estimate the answer by using the fraction method.

GIGANTIC SALE!

All items 25% off.

About how much would you save on a color T.V. which normally costs $395.95? Here are three different estimation methods that can be used.

Meaning of Percent Method	One-Percent Method	Fraction Method
25% means $25 for every $100 or $100 for every $400	1% of $400 is $4 25% is $100	25% is $\frac{1}{4}$ $\frac{1}{4}$ of $400 is $100

Use any method you wish to make your estimates.

1. SPENDLESS WAS HAVING A SALE.

   Give the approximate discount on each item.

Item	Cost	Percent Discount	Approximate Discount
Radio	$ 89.98	10%	
Calculator	$ 79.00	15%	
Watch	$ 99.00	12%	
T.V.	$ 399.98	20%	
Bicycle	$ 139.95	5%	
Sleeping bag	$ 49.95	50%	
Tennis racket	$ 59.99	25%	
Jacket	$ 69.50	30%	

Estimation

Mathematics teaching objectives:

- Use rounding and mental arithmetic in making estimates with percents.
- Review various methods for estimating percent problems.

Problem-solving skills pupils might use:

- Make reasonable estimates.
- Solve an easier but related problem.
- Solve a problem using a different method.

Materials needed:

- None

Comments and suggestions:

- Since the main purpose of this lesson is to review the percent estimation procedures introduced in the first three lessons, begin the class by giving attention to the comparison of these procedures in the chart shown at the beginning of the lesson. Present other percent problems to the class and elicit from them the information needed to make a chart showing the three different methods.
- After pupils have worked several of the problems individually, spend several minutes having pupils indicate what procedures they used in working the problems.

Answers:

Exercises

1.

Radio	$9	($9.00)	Bicycle	$ 6 to $ 7	($7.00)
Calculator	$11 to $12	($11.85)	Sleeping Bag	$24 to $25	($24.98)
Watch	$11 to $12	($11.88)	Tennis Racket	$15	($15.00)
T.V.	$78 to $80	($80.00)	Jacket	$20 to $21	($20.85)

2. 75 to 85% (82%)

Estimation (cont.)

2. Wayne had 51 magazines to sell. He sold 42 of them. Approximately what percent did he sell?

3. Most states have a sales tax. The tax normally is 4%, 5%, or 6%. About how much tax is charged on each of the following?

Cost		4% Tax	5% Tax	6% Tax
Sleeping bag	$ 28.95			
T.V.	$ 97.95			
Stereo	$209.95			

4. A certain ore contains 2% copper. How much ore is needed to obtain 24 tons of copper?

5. Mr. Stevens announced that 90% OR ABOVE would receive A's. Sandy made these scores on quizzes. On which ones did she receive an A?

Score	Did she get an A?
83 out of 100	
48 out of 50	
17 out of 25	
7 out of 10	
18 out of 20	
4 out of 5	

6. A 5% federal tax is added to the price of all airline tickets. About how much tax would be added to each of the following ticket prices?

Ticket Price	Tax
$ 79	
$199	
$397	
$205	
$525	

Answers:

Exercises (cont.)

3.

Cost	4% Tax	5% Tax	6% Tax
Sleeping Bag	$1.10 to $1.20 ($1.16)	$1.40 to $1.50 ($1.45)	$1.70 to $1.80 ($1.74)
T.V.	$3.90 to $4.00 ($3.92)	$4.80 to $5.00 ($4.90)	$5.80 to $6.00 ($5.88)
Stereo	$8.00 to $8.50 ($8.40)	$10 to $11 ($10.50)	$12 to $13 ($12.60)

4. 1200 tons

5. 83 out of 100 ⟶ no
48 out of 50 ⟶ yes
17 out of 25 ⟶ no
7 out of 10 ⟶ no
18 out of 20 ⟶ yes
4 out of 5 ⟶ no

6. $ 79 ⟶ $3.90 to $ 4.00 ($ 3.95)
$199 ⟶ $9.50 to $10.00 ($ 9.95)
$397 ⟶ $19 to $20 ($19.85)
$205 ⟶ $10 to $11 ($10.25)
$525 ⟶ $25 to $27 ($26.25)

7. 28 out of 100 ⟶ no
3 out of 10 ⟶ yes
10 out of 40 ⟶ no
25 out of 49 ⟶ yes
12 out of 50 ⟶ no
13 out of 27 ⟶ yes

8. A 30%
B 40%
C 10%
D 20%

9.

Number of Spins	A's	B's	C's	D's
100	30	40	10	20
50	15	20	5	10
75	22 or 23	30	7 or 8	15
30	9	12	3	6
500	150	200	50	100

7. Coach Johnson said, "You will receive an award if you get a hit at least 30% of the time." Who received an award?

Hits	Times at Bat	Does he get an award?
28	100	
3	10	
10	40	
25	49	
12	50	
13	27	

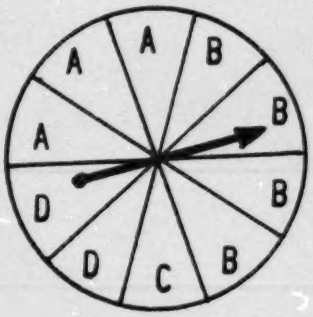
8. About what percent of the time would you expect to spin

   an A?

   a B?

   a C?

   a D?



9. Suppose you spun the spinner the number of times indicated in the chart. About how often would you expect to spin each letter?

Number of Spins	About how many A's	B's	C's	D's
100				
50				
75				
30				
500				

SHORT STORIES

Solve these problems, if possible.

Jim made 15 shots. What percent is this?

Mrs. Evans is paid a commission. One month she sold items totaling $2785. How much did she get to keep?

Wendy bought a record at a discount house. The regular price of each record was $7.45. How much did she pay?

Class Exercises

Read the above problems carefully.

1. What's unusual about each problem?

2. What information is needed to solve each of them?

3. Make up information so that each problem is complete.

4. Use an estimation method to find the approximate answers.

Exercises

Carefully read each problem below. Supply information so that each problem is complete. Then estimate the answer.

1. Steve had to pay a 15% down payment on a bicycle. About how much was his down payment?

2. Susan had a score of 80%. How many problems did she get correct?

3. Al has $91.37 in his savings account. About how much interest will he receive this year?

Short Stories

Mathematics teaching objectives:

- Use rounding and mental arithmetic in making estimates in solving percent problems.
- Search for reasonable prices for familiar everyday items.

Problem-solving skills pupils might use:

- Clarify the problem through careful reading.
- Identify problem situations in which a solution is not possible.
- Supply missing information needed for the solution of a problem.

Materials needed:

- None

Comments and suggestions:

- Pupils might supply unrealistic data for the information needed for solving the problems. Encourage them to supply realistic data by having catalogs and advertising sections from newspapers available.
- After the problems have been worked, pupils might share their problems with others. They might use their knowledge of prices in determining to what extent the supplied information was reasonable.

Answers:

Class Exercises

1. Something is missing from each "bubble" problem.
2. 1st bubble - Number of shots taken must be supplied.
   2nd bubble - Rate of commission must be supplied.
   3rd bubble - Rate of discount must be supplied.
3. Answers will vary. (Note: Insist that the supplied information be reasonable data.)
4. Answers will vary.

Exercises - (In each case, insist that efforts be made to supply reasonable data. The numerical answers for each problem will vary. The needed data for each problem is indicated below.)

1. Cost of the bicycle
2. Number of problems on the test
3. Rate of interest
4. Cost of the radio
5. Number of students in the school
6. Regular price for the motorcycle
7. Number of pages already read
8. Number of cans purchased. Cost per can. Number of cans to be purchased before discount is granted.

Extension: Answers will vary.

Short Stories (cont.)

4. Janet had to pay a 6% sales tax on a radio. About how much tax did she pay?

5. 205 students were at the game. About what percent of the student body was this?

6. Tony bought a motorcycle for 75% of its regular price. About how much did he have to pay for it?

7. Calvin has read about 50% of a novel. About how many pages does the novel have?

8. Coach Shoff will get a 20% discount by buying a certain number of cans of tennis balls. About how much money will he save?

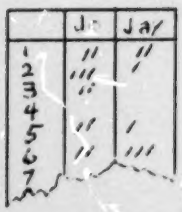
EXTENSION

Create a problem of your own, but leave out some information needed to solve it. See if a classmate can supply the needed information and estimate the answer.

Grade 8

X. PROBABILITY

X. PROBABILITY



Experimental probability provides an excellent opportunity to emphasize these problem-solving skills: collect data, make a table, use a model, make predictions and in general, guess, check and refine. As pupils suggest ways of solving the problems (or as they describe how they solved the problems) their answers can be translated into standard problem-solving phrases. For example, "Keep track of what you get" can be verbalized by the teacher as "Make a table." Specific suggestions for emphasizing the problem-solving skills are given in the comments and suggestions for each page.

Working with experimental probability in grade 8 is instructionally sound. Usually pupils are just getting ready for formal probability like "The probability of rolling a sum of 7 is $\frac{7}{36}$." Experimental probability provides some background for this work with formal probability and, more importantly, it provides experience with a method of solving problems that is very useful in the real world. To avoid preconceived notions, like "Heads are luckier than tails," the activities in this section use situations with which pupils may not be familiar. Each activity allows pupils to collect enough data to predict the outcome.

Using The Activities

With enough equipment, the entire class can complete one activity and the results can be compared and combined. Another option is to have small groups rotate through activity stations which are all on experimental probability or are on a mixture of topics.

The extra materials needed for these activities include:

- regular dice, three dice per pair of pupils.
- several coins.
- cubes with three green faces, two red faces, and one yellow face.

Pupils may need review on an efficient way to tally groups by fives.

The activities in this section can be done in any order. You may wish to provide additional activities in experimental probability. (See Mathematics Resource Project, STATISTICS AND INFORMATION GATHERING, Creative Publications.)

DICE

Evan and Ross are playing a dice game. Each time two dice are rolled, they find the sum of the dots.

Evan predicts a 10 will occur most often.
Ross predicts a 6 will occur most often.

Do you agree with either Evan or Ross? Or do you think another sum will occur more often?

1. Get two dice.

   a. Roll them 40 times altogether. Keep a record of your results.

   b. How do your results compare with your prediction?

   c. Draw a graph of your results.

2. Use three dice.

   a. Which sum would you expect to occur most often?

   b. Now roll them 40 times altogether. Keep a record of your results.

   c. How do your results compare with your prediction?

   d. Draw a graph of your results.

Dice

Mathematics teaching objectives:

- Develop probability concepts
- Draw a graph.
- Practice recording skills.

Problem-solving skills pupils *might* use:

- Collect data needed to solve the problem.
- Make and use a table.
- Look for patterns.

Materials needed:

- Three regular dice for every two people

Comments and suggestions:

- The use of home-made foam dice will make this a quieter activity.
- Pupils usually know that rolling a single die will result in each number occurring about the same number of times. When rolling two dice, pupils often are surprised to discover that some sums occur more than others.
- The activity works well with pairs of pupils, one doing the rolling and one doing the recording.
- A bar graph is best for displaying this type of data.
- Tabulate the results from the entire class. This composite often matches theoretical expectations and provides an opportunity to discuss the value of mathematics in making predictions.

Answers:

Answers will vary. Results of 40 rolls are shown. The numbers in parentheses show the theoretical numbers if two dice were rolled 36 times and if three dice were rolled 216 times.

1.

Sum	Tally	Total
2 (1)	II	2
3 (2)	I	1
4 (3)	II	2
5 (4)	III	3
6 (5)	𝍸 I	6
7 (6)	𝍸 𝍸 II	12
8 (5)	𝍸	5
9 (4)	IIII	4
10 (3)	II	2
11 (2)	II	2
12 (1)	I	1

2.

Sum	Tally	Total
3 (1)		0
4 (3)		0
5 (6)	III	3
6 (10)	III	3
7 (15)	IIII	4
8 (21)	IIII	4
9 (25)	III	3
10 (27)	𝍸 I	6
11 (27)	III	3
12 (25)	IIII	4
13 (21	II	2
14 (15)	III	3
15 (10)		0
16 (6)	IIII	4
17 (3)	I	1
18 (1)		0

ROLL THAT CUBE

Get: Colored cubes

1. Examine one of the cubes.

   a. If you were to roll only one cube 36 times, about how often would you expect to roll

      a green? a red? a yellow?

   b. Now, actually roll a cube 36 times. Keep a record of your results. How do your results compare with your answers in part a?

   c. Draw a graph of your results.

2. Use 2 cubes.

   a. If you rolled both cubes, list the color combinations that could occur.

   b. Which color combination would you expect to occur most often? least often?

   c. Roll both cubes 60 times. Keep a record of your results. How do your results compare with your answers in part b?

   d. Draw a graph of your results.

EXTENSION

Conduct a similar experiment with 3 cubes rather than with 2. Make some predictions first.

Roll That Cube

Mathematics teaching objectives:

- Develop probability concepts.
- Draw a graph.
- Practice recording skills.

Problem-solving skills pupils might use:

- Collect data needed to solve the problem.
- Make and use a table.
- Look for patterns.

Materials needed:

- Set of 12 cubes, each with 3 green faces, 2 red faces, and 1 yellow face.

Comments and suggestions:

- The use of home-made foam "dice" will make this a quieter activity.
- Because of the unusual nature of the cubes, this activity is best used with groups of pupils so the cubes can be shared. Twelve cubes should be sufficient.
- Be sure pupils predict the color or combination they think will occur most often. The situation in problem 2 is often a surprising result.
- Use a bar graph to display this data.

Answers:

Answers will vary. Sample results are shown. The numbers in parentheses show the theoretical numbers if a single cube is rolled 36 times and if two cubes are rolled 36 times.

1. a. 18, 12, 6

b.

Color	Tally	Total
Green (18)	𝍸 𝍸 𝍸 ||||	19
Red (12)	𝍸 𝍸 ||	12
Yellow (6)	𝍸	5

2. a. GG, RR, YY, GR, GY, RY

b. Answers will vary but GG is usually chosen as occurring most and YY as occurring least.

c.

Colors	Tally	Total
GG (9)	𝍸 |||	8
RR (4)	||	2
YY (1)	|	1
GR (12)	𝍸 𝍸 ||	12
GY (6)	𝍸 ||||	9
RY (4)	||||	4

A matrix can be used to show why GR occurs more often than GG.

Die 1 \ Die 2	G	G	G	R	R	Y
G	GG	GG	GG	GR	GR	GY
G	GG	GG	GG	GR	GR	GY
G	GG	GG	GG	GR	GR	GY
R	GR	GR	GR	RR	RR	RY
R	GR	GR	GR	RR	RR	RY
Y	GY	GY	GY	RY	RY	YY

Extension:

Using three colored cubes, ten different combinations are possible. The numbers in parentheses show the theoretical number for each combination if the three cubes were rolled 216 times. Many pupils would say that 3G should occur more often. As the numbers show 2G-1R occurs twice as often.
3G (27), 3R (8), 3Y (1), 2G-1R (54), 2G-1Y (27), 2R-1G (36), 2R-1Y (12), 2Y-1G (9), 2Y-1R (6), 1G-1R-1Y (36).

COIN TOSS

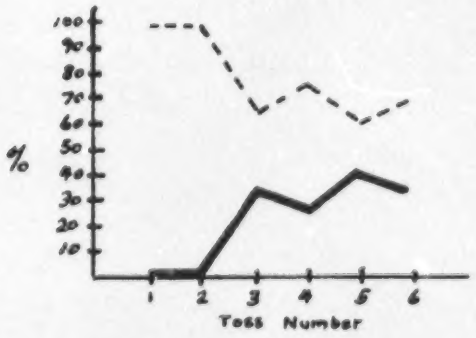
Julie tossed a coin four times. The results are summarized in the table.

Toss Number	1	2	3	4	5	6
Heads	1	2	2	3		
% Heads	100	100	67	75		
Tails	0	0	1	1		
% Tails	0	0	33	25		

1. Study the chart.
   a. What did Julie toss (heads or tails) on each of her four throws?
   b. Suppose that a "tails" was tossed on the fifth throw. How would you fill in the fifth column?
   c. Suppose Julie tossed a "heads" on the sixth throw. How would you fill in the sixth column?

2. Toss a coin 20 times. Show your results in a chart like Julie's.

3. Julie drew a graph of her results.



- - - means heads

——— means tails

Draw a graph of <u>your</u> results.

4. Suppose you tossed the coin several thousand times and drew the graph of the results. Make a prediction as to what you think the graph would begin to look like.

Coin Toss

Mathematics teaching objectives:

- Develop probability concepts.
- Make a broken-line graph.
- Practice recording skills.

Problem-solving skills pupils might use:

- Collect data needed to solve the problem.
- Make and use a table.
- Look for patterns.

Materials needed:

- Coin
- Grid paper

Comments and suggestions:

- Some pupils have difficulty getting a good flip of the coin. Shaking the coin in a small paper cup and then turning the cup over is a good alternative.
- The line graph usually is used for situations where the data are continuous, not discrete as in this case. A bar graph is better for discrete cases. But this activity is showing how the two percents converge toward 50%. "Poetic license" allows us to use the line graph for this situation.

Answers:

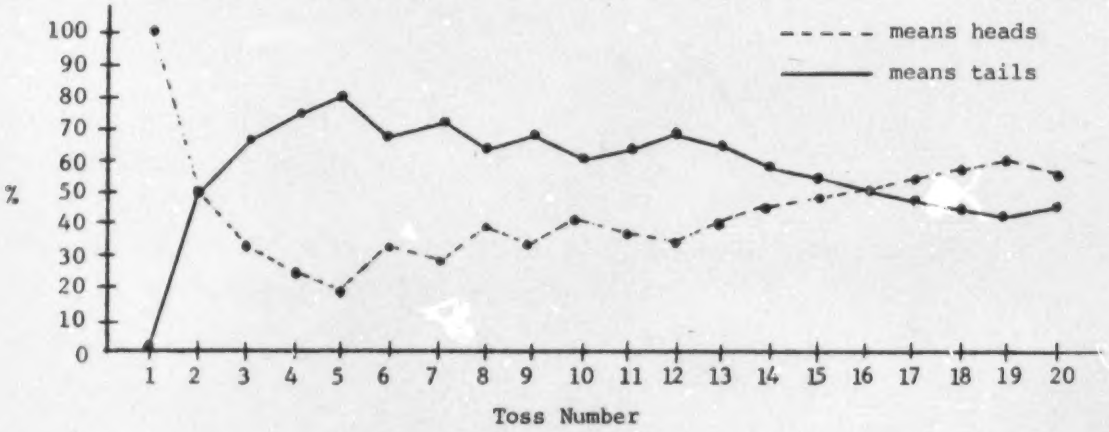
1. a. Toss 1, heads; Toss 2, heads; Toss 3, tails; Toss 4, heads

   b. 3 heads; 60% heads; 2 tails; 40% tails

   c. 4 heads; 67% heads; 2 tails; 33% tails

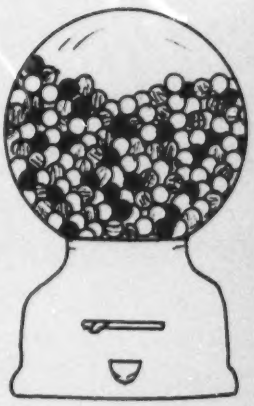
2, 3. Answers will vary. A sample of 20 tosses is shown.

Toss Number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Heads	1	1	1	1	1	2	2	3	3	4	4	4	5	6	7	8	9	10	11	11
% Heads	100	50	33	25	20	33	29	38	33	40	36	33	38	43	47	50	53	56	58	55
Tails	0	1	2	3	4	4	5	5	6	6	7	8	8	8	8	8	8	8	8	9
% Tails	0	50	67	75	80	67	71	62	67	60	64	67	62	57	53	50	47	44	42	45



4. The trend should be toward 50% heads and 50% tails.

THE GUM MACHINE



Karen and Diane are having an argument. They both know that the machine is loaded with gum balls of 6 different colors.

Karen feels lucky. She thinks she can get each color by using only 6 pennies. Diane feels it will take at least 15.

Let's assume that the machine is loaded with an equal number of each color. How many pennies do you think are needed to get one of each color? Do you agree with either Karen or Diane?

Here is a way to predict the average number of pennies needed to get a gum ball of each color. Follow these steps:

1. Get a die with each of the 6 faces a different color. (Or use an ordinary die on which 1 represents red, 2 represents blue, etc.)

2. Toss the die until one of each color has appeared. How many tosses (pennies) did it take?

3. Repeat the experiment four more times.

4. Use the results of your 5 experiments to find the average number of pennies needed.

5. Was your estimate better than Karen's? Diane's?

EXTENSION

Suppose the machine is loaded with gum balls of 8 different colors. How many pennies do you think would be needed to get one of each color? Set up an experiment to check your prediction.

The Gum Machine

Mathematics teaching objectives:

- Develop probability concepts.
- Practice recording skills.

Problem-solving skills pupils might use:

- Collect data needed to solve the problem.
- Make and use a table.

Materials needed:

- One die per pupil

Comments and suggestions:

- Point out that after the removal of the first gum ball the color for the next gum ball is no longer equally likely. Discuss the reasonableness of the assumption. For all practical purposes, as long as the number of gum balls in the machine is large, the chances for getting a certain color is essentially the same.
- This simulation also can be done using pieces of colored paper. Put six different colors in a container. Have pupils draw a piece, without looking, and record the color. Replace the piece; mix the pieces up; and draw again. Continue until all six colors have been drawn.
- The extension can be simulated using an octahedral die, eight pieces of paper, a spinner with eight outcomes, or eight cards marked with the digits 1-8 representing the eight colors.

Answers:

4. Answers will vary. Five experiments, using a die, produced these results: 14, 19, 18, 8 and 8 for an average of 13.4 . Pupils should see that this really means 14 pennies. The theoretical number is 14.7 or 15 pennies.

Extension:

Answers will vary. Five experiments, using a die, produced these results: 14, 30, 24, 13 and 20 for an average of 20.2 . Pupils should see that this really means 21 pennies. The theoretical number is 21.74 or 22 pennies.

Grade 8

XI. CHALLENGES

XI. CHALLENGES

The activities in the Getting Started section were very teacher-directed and pupils were encouraged to use (although not completely restricted to) one problem-solving skill at a time. The challenge problems in this section leave the choice of the problem-solving method up to the pupil. The intention is to allow for and encourage individual differences, creativity, and cooperation.

Let's look at how one challenge problem, "Magic Circles" (page 259), can be used in the classroom. As you read the example, notice how the teacher does not structure or direct the methods pupils use, but that the teacher does have these important functions:

- to help pupils understand the problem.
- to listen if pupils want to discuss their strategies.
- to praise and encourage pupils in their attempts, successful or not.
- to facilitate discussion of the problem and sharing of the strategies.
- to give hints or ask questions, if necessary.
- to summarize or emphasize methods of solution after pupils have solved the problem.

As we look into Mr. Lane's classroom, the page, "Magic Circles," has been distributed. Pupils are accustomed to seeing a different challenge each week and they know they will be given some time during the week to work on the problem.

Mr. L: Here is the challenge problem for the week. I'll let you look at the problem, then we'll discuss it to be sure we all understand it. (Waits.) Merlin, you look puzzled.

Merlin: Can I use some of the numbers more than once?

Mr. L: No. You must use each of the numbers 2 to 10 once and only once. Any other questions?

Jane: I know what the outside numbers have to be. But I don't know how to go about finding the numbers for the smaller circles.

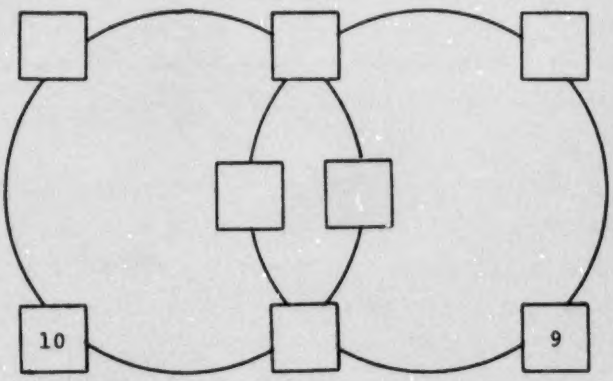
Mr. L: It wouldn't be a challenge if you knew right away, would it? What kinds of things can you try?

Jane: (Looking up at the posted list of problem-solving skills.) I'll try guessing.

Mr. L: Good idea. (In a quiet, side conversation with Peggy.) Peggy, I notice that you've broken the problem down into smaller parts. Why did you do that?

Peggy: By breaking it down, I see that these two circles can have only two numbers in common. This way, I think I'll be able to eliminate certain possibilities.

Peggy's Work



Mr. L: That's an excellent strategy to use. (In another side conversation with Roy.) Roy, you seem to be frustrated with the problem. I was too, until I found this number. (Shows him where to put a number in an "inside" circle.)

Mr. L (To the whole class.) I noticed that Roy was using markers with numbers on them. This is a good idea since it eliminates much erasing. Tomorrow, perhaps you'll want to make some of your own markers to help you in your search for a solution to the problem.

The above approach to challenge problems also gives opportunities for practicing the following problem-solving skills:

- State the problem in your own words.
- Clarify the problem through careful reading and by asking questions.
- Share data and the results with other interested persons.
- Listen to persons who have relevant knowledge and experiences to share.
- Study the solution process.
- Invent new problems by varying an old one.

THE TEACHER MUST BE AN ACTIVE, ENTHUSIASTIC SUPPORTER OF PROBLEM SOLVING.

Using The Activities

Sixteen varied challenge problems are provided. Some teachers give them as a "challenge of the week" or as a Friday activity. On the day the challenge is given out, time should be spent on getting acquainted with the problem. On following days, a few minutes can be devoted to pupil progress reports. If there is little sign of progress, you can provide some direction by asking a key question or suggestion a different strategy. At appropriate times, the activity can be summarized by a class discussion of strategies used and some problem extensions.

One Plan For Using A Challenge (over a period of 1 or 2 weeks)

First day -

- Give out the challenge. (Possibly near the end of the period.)
- Let pupils read written directions and possibly discuss with a classmate.
- Clarify any vocabulary which seems to be causing difficulty. Ask a few probing questions to see if they have enough understanding to get started.
- Remind them that during a later math class, time will be used to look at the problem again.

Later in the week -

- Have pupils share their ideas.
- Identify the problem-solving skills suggested by these ideas.
- Conduct a brainstorming session if pupils do not seem to know how to get started.
- Suggest alternative strategies they might try.
- Give an extension to those pupils who have completed the challenge.

On a subsequent day -

- Allow some time for individuals (or small groups) to work on the challenge. Observe and encourage pupils in their attempts.
- Try a strategy along with the pupils (if pupils seem to have given up.)

Last day -

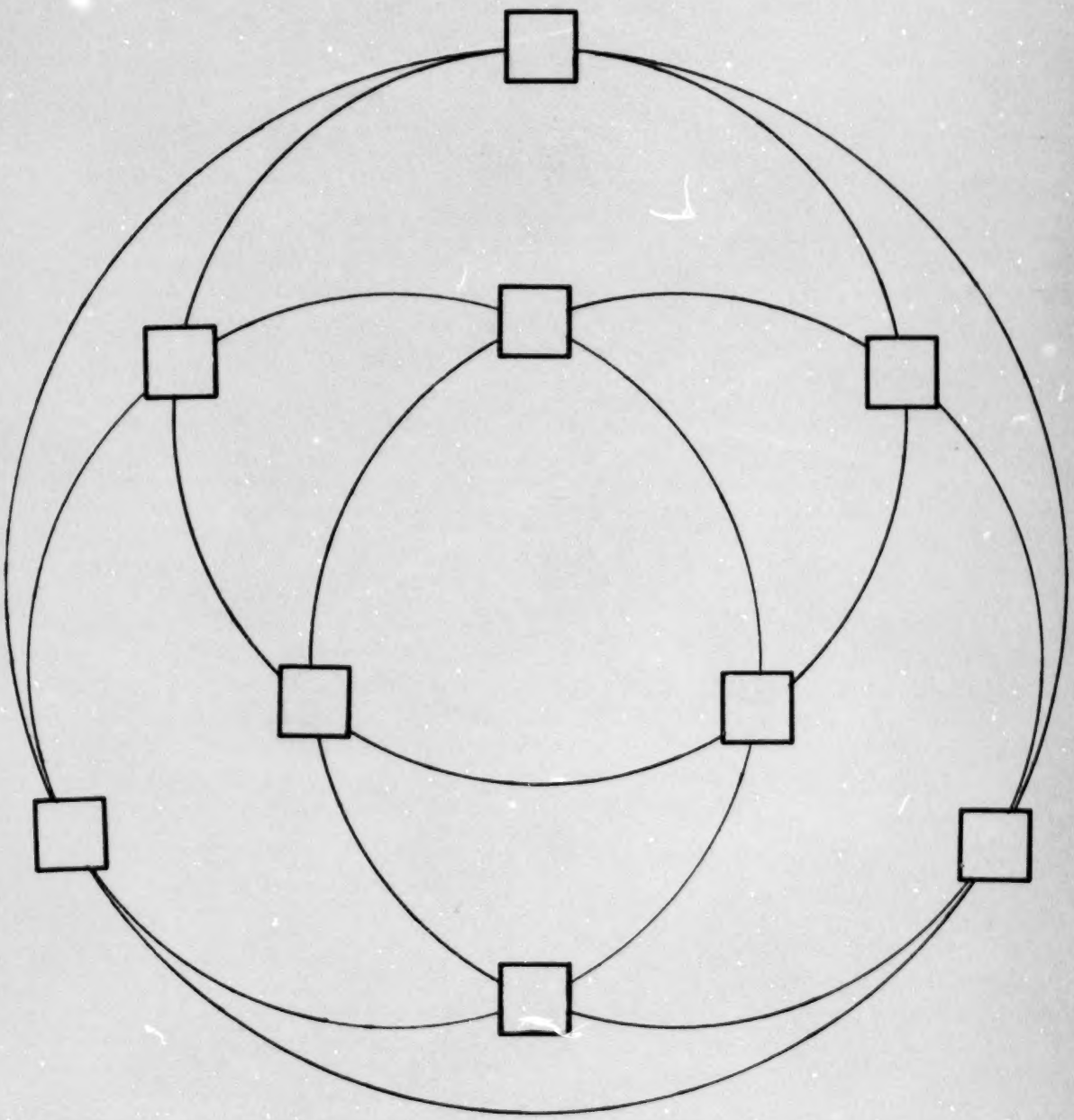
- Conduct a session where pupils can present the unsuccessful as well as the successful strategies they used.
- Possibly practice a problem-solving skill that is giving pupils difficulty; e.g., recording attempts, making a systematic list, or checking solutions.

Key problem-solving strategies pupils have used in solving the problems are given in the comments for each problem. Your pupils might have additional ways of solving the problems.

A challenge problem for the teacher: Keep the quick problem solver from telling answers to classmates.

MAGIC CIRCLES

1. Place the numbers 2, 3, 4, 5, 6, 7, 8, 9, 10 in the boxes so the sum around each of the four circles is 27.



2. Use the same rules as before. This time, use 1, 2, 3, 4, 5, 6, 7, 8, 9. The magic sum should be 24.

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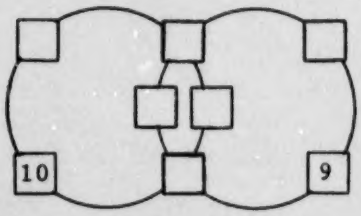
Magic Circles

Problem-solving skills pupils might use:

- Guess and check.
- Eliminate possibilities,
- Breaking the problem into smaller parts.

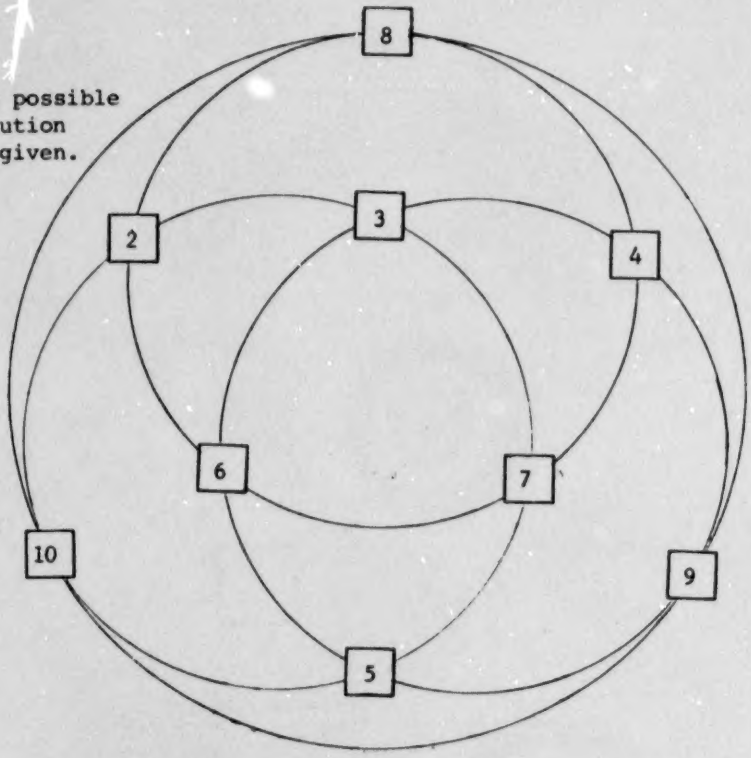
Comments and suggestions:

- Some pupils may need a hint if they become discouraged. Perhaps one or two of the inside numbers could be given.
- One strategy is to focus on two of the smaller circles... for example, the one that uses "10" and the one that uses "9." These circles intersect in exactly two points. Thus, in each of these circles, only two of the five numbers can be the same.



Answers:

1. One possible solution is given.



2. Not possible.

USE ALL NINE

For each problem below:

- Fill in the blanks with the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 to make a true statement.
- Use each number only once.
- Some blanks are filled in to help you.

1. (___ + ___) - 9 = 1
   (___ - ___) x 1 = 1
   (___ - ___) ÷ 2 = 1

2. (9 + ___) ÷ ___ = 2
   (___ - 5) + ___ = 2
   (___ x ___) - 4 = 2

3. (___ + 7) ÷ ___ = 3
   (8 - ___) - ___ = 3
   (___ - 6) ÷ ___ = 3

4. (9 + ___) ÷ ___ = 4
   5 - (___ - ___) = 4
   (8 x ___) ÷ ___ = 4

5. (___ + 9) - ___ = 5
   (___ x 6) - ___ = 5
   (___ ÷ 4) + ___ = 5

6. ___ - (___ - 5) = 6
   ___ x (___ ÷ 4) = 6
   ___ - (___ + 1) = 6

7. 4 + (___ ÷ ___) = 7
   (___ + 6) - ___ = 7
   ___ + (___ x 2) = 7

8. ___ + (___ ÷ 2) = 8
   (___ + 3) x ___ = 8
   (9 - ___) + ___ = 8

9. (2 x ___) - ___ = 9
   (___ ÷ 3) + ___ = 9
   ___ + (___ x 1) = 9

10. (___ + ___) - ___ = 10
    ___ + (9 ÷ ___) = 10
    ___ + (___ - ___) = 10

Extension:

Create your own statements so each equals zero.
Remember to use the numbers 1 to 9 only once.

___ ___ ___ = 0
___ ___ ___ = 0
___ ___ ___ = 0

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Use All Nine

Problem-solving skills pupils might use:

- Guess and check.
- Eliminate possibilities.

Comments and suggestions:

- The problems are not arranged according to difficulty. If pupils get frustrated with a particular problem, have them go on to the next one.
- It's possible that the use of parentheses will need to be reviewed.
- It is suggested that you work through the first problem with the class. Emphasize that each number is to be used only once.
- A poster could be prepared on which pupils place the problems they created for the "extension."

Answers:

Only one solution is given to each problem. Some of them, however, have multiple solutions.

1. $(7 + 3) - 9 = 1$
   $(5 - 4) \times 1 = 1$
   $(8 - 6) \div 2 = 1$

2. $(9 + 7) \div 8 = 2$
   $(6 - 5) + 1 = 2$
   $(2 \times 3) - 4 = 2$

3. $(5 + 7) \div 4 = 3$
   $(8 - 2) - 3 = 3$
   $(9 - 6) \div 1 = 3$

4. $(9 + 7) \div 4 = 4$
   $5 - (2 - 1) = 4$
   $(8 \times 3) \div 6 = 4$

5. $(1 + 9) - 5 = 5$
   $(2 \times 6) - 7 = 5$
   $(8 \div 4) + 3 = 5$

6. $7 \times (6 - 5) = 6$
   $3 \times (8 \div 4) = 6$
   $9 - (2 + 1) = 6$

7. $4 + (9 \div 3) = 7$
   $(8 + 6) - 7 = 7$
   $5 + (1 \times 2) = 7$

8. $6 + (4 \div 2) = 8$
   $(5 + 3) \times 1 = 8$
   $(9 - 8) + 7 = 8$

9. $(2 \times 8) - 7 = 9$
   $(9 \div 3) + 6 = 9$
   $5 + (4 \times 1) = 9$

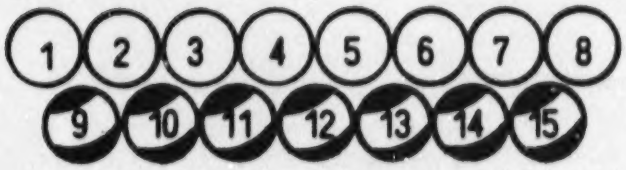
10. $(6 + 5) - 1 = 10$
    $7 + (9 \div 3) = 10$
    $4 + (8 - 2) = 10$

Extension:

One possible solution is given.

$(2 \times 3) - 6 = 0$
$(7 + 1) - 8 = 0$
$(4 + 5) - 9 = 0$

POOL



Pool balls are solid (1 to 8) or striped (9 to 15).
The colors are:

	yellow	(1 and 9)
	blue	(2 and 10)
	red	(3 and 11)
	purple	(4 and 12)
	orange	(5 and 13)
	green	(6 and 14)
	violet	(7 and 15)
and	black	(8)

Sally and Pat are playing rotation pool. In rotation pool, each girl gets to count the numbers on any balls that she sinks. At the end of the game, the girl with the highest total wins.

1. The first game ended in a tie. But Sally sank only 5 balls. Show how this is possible in several different ways.

2. Pat won the second game. She beat Sally by 2 points. She sank both blues, both reds, one green, and one orange. Was it possible for her to win by sinking only one more ball? If so, which one?

3. Sally won the third game. She had 75 points and sank the 15-ball last. Pat sank only 5 balls--two were striped, one was the 3-ball, and none were blue or purple.
Who sank the 8-ball? Show how you arrived at your answer.

4. Who won the fourth game? You decide by making up your own clues. Then, give the problem to someone else to solve.

Pool

Problem-solving skills pupils might use:

- Guess and check.
- Eliminate possibilities.
- Use a systematic list.

Comments and suggestions:

- The game of rotation pool may need to be explained to the class.
- Perhaps someone in the class has a pool table and could bring in the set of balls--or at least a "stripe" and a "solid."
- If a set of pool balls is available, they could be used effectively to visually demonstrate the solution process.
- You might suggest that pupils cut out and label little pieces of paper to represent the balls.

Answers:

1. The sum of all fifteen balls is 120. Thus, in order to tie, each girl has to have a sum of 60. There are seven possible ways for Sally to get a sum of 60 with five balls.

   $15 + 14 + 13 + 12 + 6 = 60$
   $15 + 14 + 13 + 11 + 7 = 60$
   $15 + 14 + 13 + 10 + 8 = 60$
   $15 + 14 + 12 + 11 + 8 = 60$
   $15 + 14 + 12 + 10 + 9 = 60$
   $15 + 13 + 12 + 11 + 9 = 60$
   $14 + 13 + 12 + 11 + 10 = 60$

2. The final score has to be 61 to 59 in favor of Sally. It is possible for Sally to win by sinking only one more ball--the 8-ball. In order to do this, she must sink the 14 (green) and the 13 (orange).

3. The sum of Pat's five balls has to be 45; but one of them is the 3. Therefore, the sum of four balls is 42. Since two are striped, the only possibility for Pat is 3 - 7 - 8 - 13 - 14. Pat sank the 8-ball.

4. Answers will vary.

Problem 1

An artist, carpenter, dancer, and grocer are named Alice, Cory, Dan, and Gloria. Use the clues to help you match the names with the occupations.

1. Only one person's name begins with the same letter as the occupation.
2. Dan is richer than either the carpenter or Gloria.
3. The grocer is younger than Alice.
4. Cory is the oldest and is a good friend of the carpenter.
5. Dan is older than Alice.

Problem 2

Olson, Smith, Hayward, and Duncan had dinner together one evening. When they left the restaurant each of them, by mistake, was wearing the coat belonging to someone else in the party and the hat of yet someone else. Smith took Olson's hat, and the man who took Duncan's hat took Hayward's coat. Which hat and coat did each of the men take?

Logical Thinking

Problem-solving skills pupils might use:

- Use a chart to keep track of information.
- Eliminate possibilities.

Comments and suggestions:

- In solving logic problems, one needs to find a system for keeping track of and sorting out information. A chart often is the best way to accomplish this. Pupils may need to be shown how such a chart can be constructed.
- Pupils should be encouraged to reread the problem each time new information is uncovered.

Answers:

Problem 1

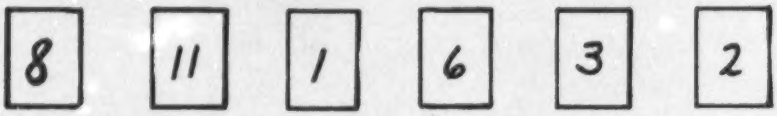
	Alice	Cory	Dan	Gloria
Artist	No	No		No
Carpenter		No	No	No
Dancer	No		No	No
Grocer	No	No	No	

Problem 2

	Olson		Smith		Hayward		Duncan	
	Coat	Hat	Coat	Hat	Coat	Hat	Coat	Hat
Olson	No	No	No	No		No	No	
Smith	No		No	No	No	No		No
Hayward		No	No		No	No	No	No
Duncan	No	No		No	No		No	No

A GAME OF CARDS

Kim has a special deck of cards. After shuffling the cards she turned six of them face up on the table.



Kim explained the game this way:

- Use each of the first five numbers in any order.
- Use any operations you want.
- Your answer must be the last card turned up (in this case, 2).

Here are two different solutions. Study them to make sure they're correct.

$$\frac{11 + 1}{3} - (8 - 6) = 2$$

$$\left(8 - \frac{1 + 11}{6}\right) \div 3 = 2$$

See if you can find another solution.

Kim dealt out five more hands. See how many different solutions you can find for each of them. Remember to express the results in correct mathematical form.

1. 3, 7, 1, 8, 13 _____ Answer is 6
2. 10, 15, 2, 10, 7 _____ Answer is 11
3. 4, 16, 7, 8, 9 _____ Answer is 1
4. 12, 7, 9, 7, 1 _____ Answer is 13

*5. 4, 25, 10, 22, 4 _____ Answer is 4

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A Game Of Cards

Problem-solving skills pupils might use:

- Guess and check.
- Use appropriate notation.

Comments and suggestions:

- The class may wish to make their own sets of cards and create and solve their own problems.
- Emphasize the importance of using proper mathematical symbolism. (See the use of parentheses and brackets in the answers below.)
- You may wish to have the pupils make a bulletin board display of different solutions.
- You may wish to allow pupils to use exponential, factorial or square root notation in their solutions.

Answers:

Answers will vary. A possible solution for each problem is given.

1. $8 - \left[13 - (7 + 3) - 1\right] = 6$

2. $(10 \div 10) \left(\frac{15 + 7}{2} \right) = 11$ Note the use of the multiplication property of 1.

3. $\left[(8 \div \frac{16}{4}) + 7 \right] \div 9 = 1$

4. $(7 - 7) \times 9 + 12 + 1 = 13$ Note the use of the multiplication property of 0 .

* 5. $(25 \div 10) + (22 \div 4) - 4 = 4$

A solution that uses an exponent is:

$$\left(\frac{4}{4} \right)^{10} + (25 - 22) = 4$$

A solution that uses square root is:

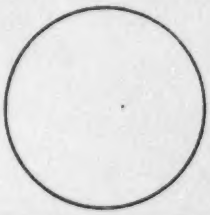
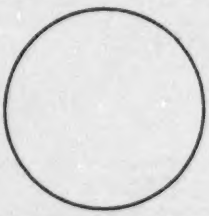
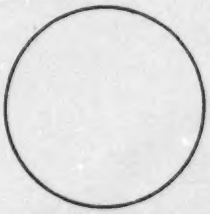
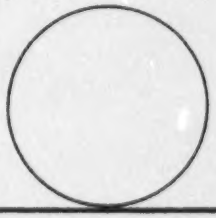
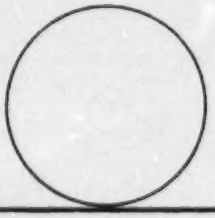
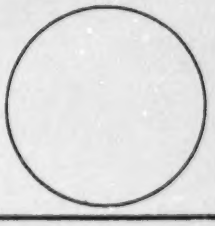
$$\frac{10 + 22 + 4}{4 + \sqrt{25}} = 4$$

PIZZA PUZZLES

These rules must be followed when cutting the pizzas:

- . All cuts must be straight.
- . All cuts must be from edge-to-edge.
- . All pieces do not have to be the same size.

1. Suppose you made just four cuts. Show how to get the following number of pieces.

5 pieces	6 pieces	7 pieces	8 pieces
			
9 pieces	**10 pieces**	**11 pieces**	**12 pieces (?)**
			

2. The fewest number of pieces resulting from four cuts is 5.

   a. Complete the table which shows the fewest pieces resulting from different numbers of cuts.

Cuts	1	2	3	4	5	6	10	12
Fewest number of pieces				5				

   b. What is the fewest number of pieces that will result from 100 cuts?

3. The greatest number of pieces resulting from four cuts is 11.

   a. Complete the "greatest number of pieces" table.

Cuts	1	2	3	4	5	6	10	12
Greatest number of pieces				11				

   * b. What is the greatest number of pieces that will result from 100 cuts?

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Pizza Puzzles

Problem-solving skills pupils might use:

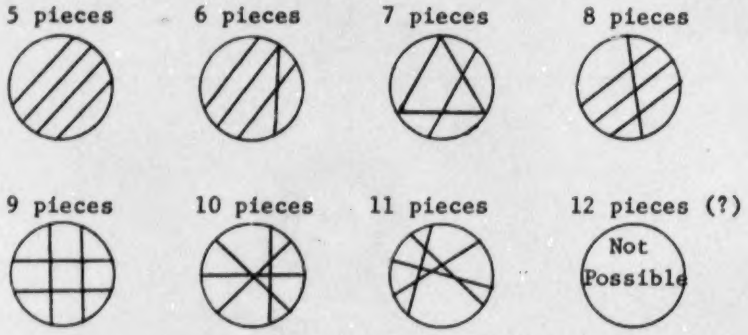
- Guess and check.
- Look for patterns.
- Use of drawings.

Comments and suggestions:

- Spaghetti or toothpicks could be used to represent the cuts.
- An extension to the problem might be called "the sloppy pizza problem." In this case, after a cut is made the resulting pieces are stacked on top of one another. Then another cut is made and another stacking takes place and this process is continued.

Answers:

1. Answers will vary. Here are some possibilities:



2. a.

Cuts	1	2	3	4	5	6	10	12
Fewest number of pieces	2	3	4	5	6	7	11	13

b. 101 Note that the number of pieces is equal to one more than the number of cuts ($P = C + 1$).

3. a.

Cuts	1	2	3	4	5	6	10	12
Greatest number of pieces	2	4	7	11	16	22	56	79

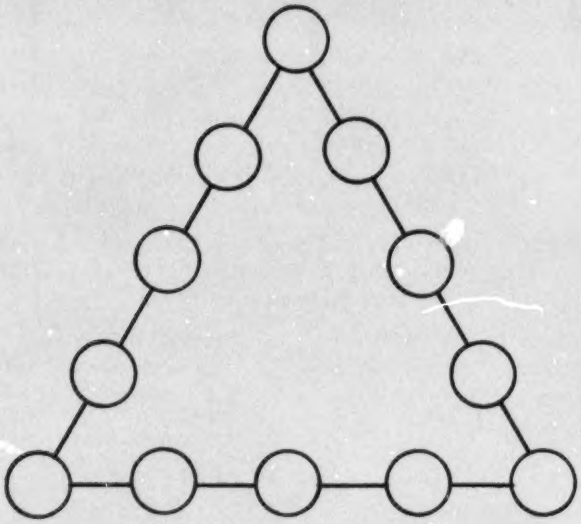
Note that the "greatest number of pieces" jumps by 2, then by 3, then by 4, etc. Some pupils will also note the following pattern: In order to find the number of pieces, say for 5 cuts, add 5 to the previous number of pieces (5 + 11 = 16).

* b. 5051 Some pupils may be tempted to continue the pattern up to 100 cuts. Instead they should be encouraged to look for a pattern that does not rely on the previous entry. One formula is:

$$p = \frac{c\,(c+1)}{2} + 1$$

(Note that this is 1 more than the sum of the first n counting numbers.)

MAGIC TRIANGLE



1. Use the numbers 1 to 12 to make each side of the triangle add to the same sum. Keep a record of your results.

   a. Put 1, 2, and 3 at the corners. Put 4, 5, and 6 in the middle of each side so each partial sum is the same.

   Now, place the remaining numbers so each sum is the same.

   What is the "magic" sum for each side? _____

   b. Put 1, 2, and 3 at the corners. Put 7, 8, and 9 in the middle so each partial sum is the same. Now, place the remaining numbers.

   What is the "magic" sum for each side? _____

   c. Repeat using 1, 2, and 3 at the corners and 10, 11, and 12 in the middle. What is the "magic" sum? _____

2. The "magic" sum using 1, 2, 3 at the corners is the lowest "magic" sum possible. Find the highest "magic" sum possible. Use one of the triangles on the next page to record your result.

3. Find as many other "magic" sums as you can that are in between the lowest and the highest. Record your results. (There are at least two more possibilities.)

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Magic Triangles

Problem-solving skills pupils might use:

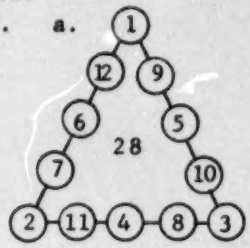
- Guess and check.
- Discover other solutions by studying ones already found.
- Look for patterns.

Comments and suggestions:

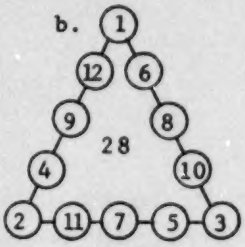
- Markers with the numbers 1 to 12 may be used to try possibilities. This procedure avoids a lot of erasing.
- Once pupils have discovered the lowest and highest sums, encourage them to study the placement of the numbers. This may provide clues that will help them find other solutions.
- You may wish to ditto more magic triangle sheets.

Answers:

1. a.



b.



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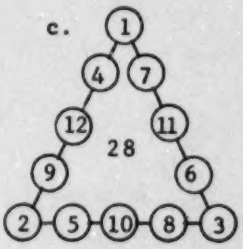

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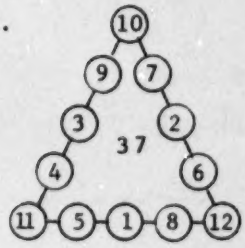

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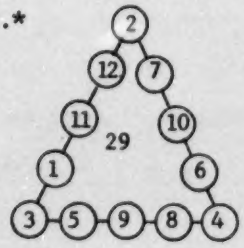
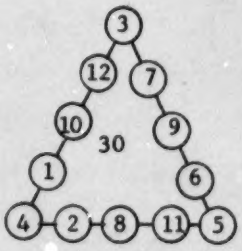
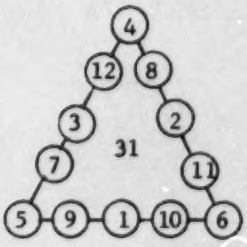
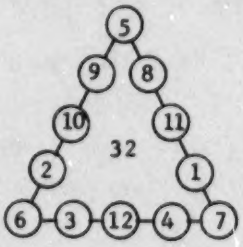
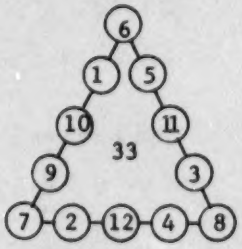
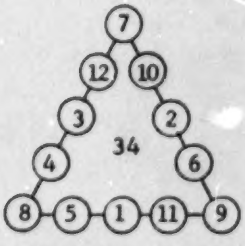


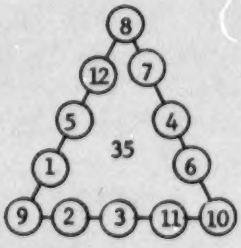


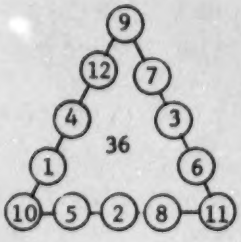






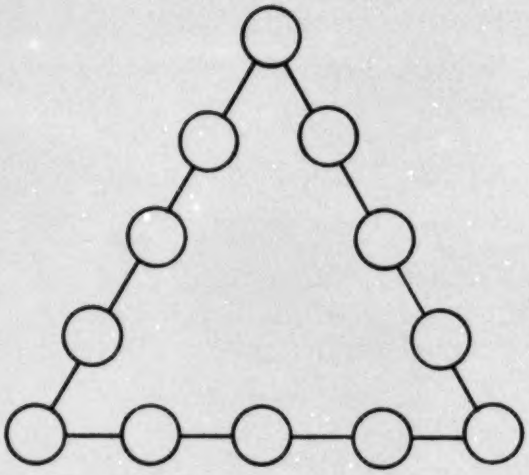


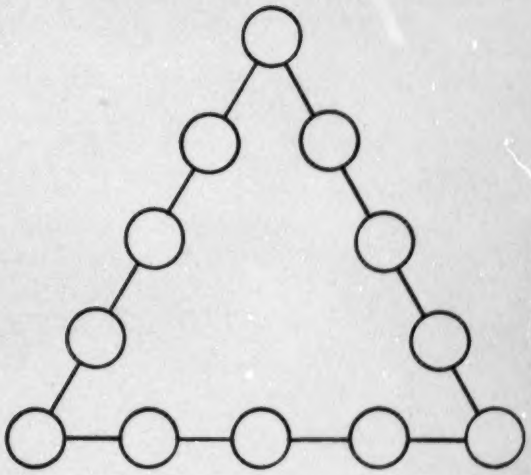


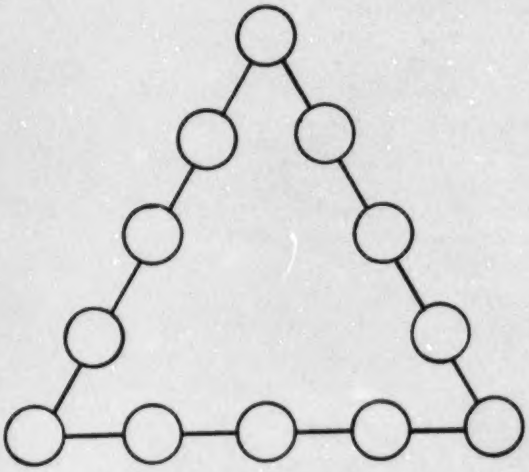


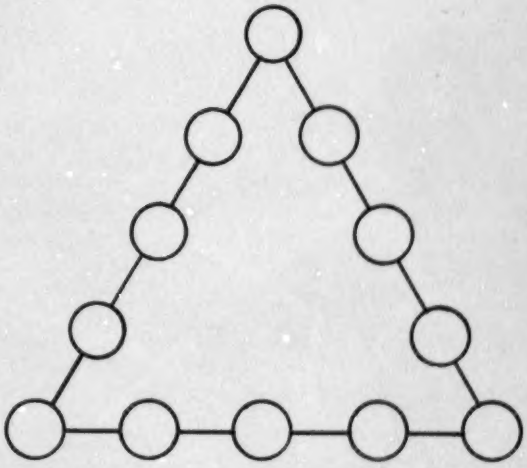
*A possible solution for each "magic sum" is given.

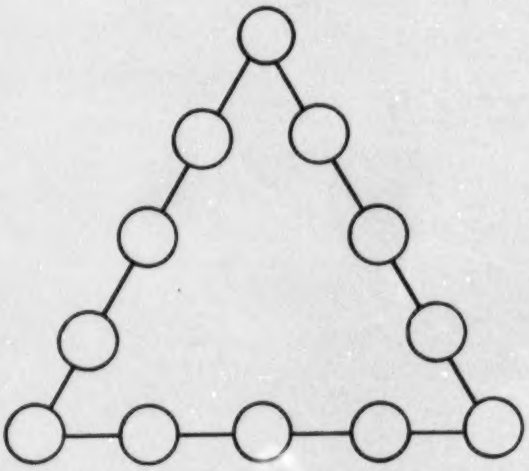
Magic Triangle (cont.)

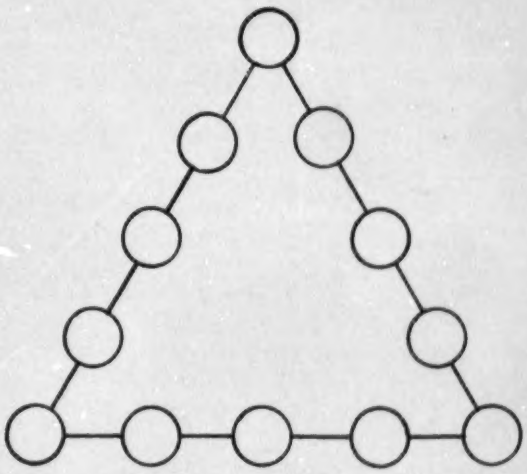












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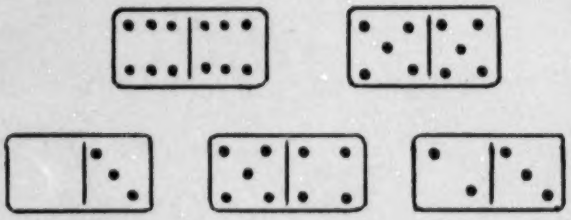
DOMINOES

A normal set of dominoes has 28 pieces.

The set contains double dominoes like 6-6, 5-5, etc.

The set also contains pieces like 0-3, 5-4, 2-3, etc.



1. Show that there are 28 pieces altogether.

2. How many dominoes are in a set if the highest double is
   a. 7-7? b. 8-8? c. 12-12?

3. Which of these formulas might have helped you find the solution to these problems? (D is the number on each side of the highest double.)

   a. $P = 8 \cdot D + 2$ b. $P = \frac{(D+1)\cdot(D+2)}{2}$ c. $P = \frac{D^2+3\cdot D+2}{2}$

4. What is the highest double in a set containing
   a. 66 dominoes? b. 231 dominoes? c. 861 dominoes?

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Dominoes

Problem-solving skills pupils might use:

- Make a systematic list.
- Look for patterns.

Comments and suggestions:

- An actual set of dominoes would be of help to pupils who are unfamiliar with the various pieces.

Answers

1. 6-6 5-5 4-4 3-3 2-2 1-1 0-0
   6-5 5-4 4-3 3-2 2-1 1-0
   6-4 5-3 4-2 3-1 2-0
   6-3 5-2 4-1 3-0
   6-2 5-1 4-0
   6-1 5-0
   6-0

   Note that this list shows that there are:
   $7 + 6 + 5 + 4 + 3 + 2 + 1$ or 28 dominoes.

   Alternate solution:
   There are seven choices for each side of a dominoe (0,1,2,3,4,5,6). Therefore, 7 x 7 or 49 pairs are possible. But seven of these are doubles, so 49 - 7 or 42 possible pairs remain. But half of the remaining 42 are duplicates (5-2, 2-5, etc.). So $\frac{42}{2}$ or 21 dominoes remain. 21 + 7(doubles) = 28.

2. a. 36 b. 45 c. 91
   These results can be arrived at by constructing and extending a table like this one:

Highest Double	0	1	2	3	4	5	6	7	8
Number of Pieces	1	3	6	10	25	21	28	36	45

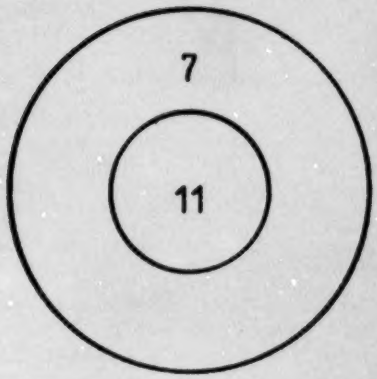
   Note that the number of pieces increases by 2, then by 3, then by 4, etc.

3. b and c both will give correct answers.

4. a. 10 b. 20 c. 40

GAME OF DARTS

Doug and Bill made up a new dart game. Each time you hit the bulls-eye you score 11 points. A dart in the outer ring counts 7 points.



1. In the first game, Bill threw 5 darts. All of them hit the target. His score was 43.

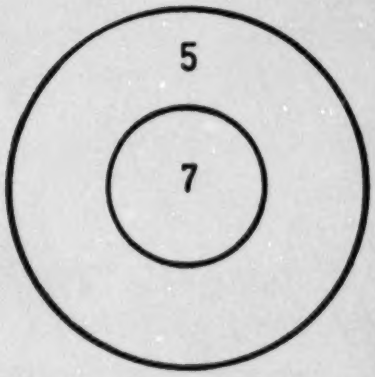
   Show how this is possible.

2. In the next game, Doug scored 32 points to Bill's 36. If all the darts hit the target, how many did each boy throw?

   Show how each of these scores is possible.

3. In the third game, Doug said his final score was 41. He did not say how many darts he threw. Do you think he may have made a mistake in adding? Explain.

4. On another dart board you can score either 5 or 7 points with one dart. It is impossible to get a total of 6, 8, or 9 by throwing darts at the target. Altogether, there are 12 impossible totals. See if you can find them all.



EXTENSION

What is the highest total which is impossible to get on the 7-11 dart board?

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Game of Darts

Problem-solving skills pupils might use:

- Make systematic lists.
- Look for patterns.

Comments and suggestions:

- Pupils may need to be shown how lists can be helpful in organizing the information.

  Multiples of 7 - 7 14 21 28 35 42 49 56 ...

  Multiples of 11 - 11 22 33 44 55 66 77 88 ...

  Possible scores can now be found by systematically combining these lists. For example, first add 7 to each of the multiples of 11, then add 14 to each of the multiples of 11, etc.

Answers:

1. $3 \times 7 + 2 \times 11$ (3 sevens and 2 elevens)

2. Doug: $3 \times 7 + 1 \times 11 = 32$
   Bill: $2 \times 7 + 2 \times 11 = 36$

3. Yes, he made a mistake. It is impossible to have a combination of 7's and 11's with a result of 41.

4. Impossible totals are 1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18, 23. When we find 5 consecutive totals which are "possible" we know that all succeeding totals also are possible.

Extension:

With a 7-11 dart board the highest impossible total is 59. In general, if two numbers (m and n) are relatively prime, then the highest impossible total is $m \times n - (m + n)$.

As a follow-up: Pupils could check this formula by experimenting with other pairs of relatively prime numbers.

HAPPY NUMBERS

19 is a happy number. Let's see why.

$$
\begin{aligned}
19 &\longrightarrow 1^2 + 9^2 &&= 82\\
82 &\longrightarrow 8^2 + 2^2 &&= 68\\
68 &\longrightarrow 6^2 + 8^2 &&= 100\\
100 &\longrightarrow 1^2 + 0^2 + 0^2 &&= 1\\
1 &\longrightarrow 1^2 &&= 1\\
1 &\longrightarrow 1^2 &&= 1\\
&\vdots
\end{aligned}
$$

Do you see how 19 eventually results in 1?

Do you think this will work for other numbers? If it does, then the number you started with is said to be HAPPY.

If you are 13 years old, then your age is HAPPY. Show that this is true.

Not all numbers are HAPPY. In fact, there are only 19 numbers less than 100 that are HAPPY.

See how many of these you can find.

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Happy Numbers

Problem-solving skills pupils might use:

- Look for patterns.
- Discover other solutions by studying ones already found.

Comments and suggestions:

- Pupils will discover that numbers either converge to 1 or to 89.
- Perhaps it would be good if an example was given that converges to 89:

$$16 \longrightarrow 1^2 + 6^2 = 37$$
$$37 \longrightarrow 3^2 + 7^2 = 58$$
$$58 \longrightarrow 5^2 + 8^2 = 89$$
$$89 \longrightarrow 8^2 + 9^2 = 145$$
$$145 \longrightarrow 1^2 + 4^2 + 5^2 = 42$$
$$42 \longrightarrow 4^2 + 2^2 = 20$$
$$20 \longrightarrow 2^2 + 0^2 = 4$$
$$4 \longrightarrow 4^2 = 16$$
$$16 \longrightarrow 1^2 + 6^2 = 37$$
$$37 \longrightarrow 3^2 + 7^2 = 58$$
$$58 \longrightarrow 5^2 + 8^2 = 89$$

- Encourage pupils to keep a complete record of their "chains" like the one above. There is much information that can be obtained by examining their structures.
- Pupils soon will discover that most HAPPY numbers come in pairs. That is, if 13 is HAPPY, so is 31.

Answers:

13 is a HAPPY number.

$$13 \longrightarrow 1^2 + 3^2 = 10$$
$$10 \longrightarrow 1^2 + 0^2 = 1$$

There are 19 HAPPY numbers less than 100:

1, 7, 10, 13, 19, 23, 28, 31, 32, 44, 49, 68, 70, 79, 82, 86, 91, 94, 97

THE LAST DIGIT

5^{1000} is a huge number.

In fact, the answer has close to 700 digits.

Do you think it is possible to determine the last digit of
5^{1000}? or 2^{1000}? or ?

Find the last digit of each of these number grids.

1. 1^{1000}
2. 2^{1000} (The last digit is not 4.)
3. 3^{1000}
4. 4^{1000}
5. 5^{1000}
6. 6^{1000}
7. 7^{1000}
8. 8^{1000}
9. 9^{1000}
10. 10^{1000}
11. 11^{1000}
12. 12^{1000}
13. 13^{1000}
14. 14^{1000}
15. 15^{1000}
16. 79^{1000}
17. 273^{1000}
18. 1000^{1000}

BONUS

What are the last two digits of 6^{1000}?

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The Last Digit

Problem-solving skills pupils might use:

- Make a systematic list.
- Look for patterns.

Comments and suggestions:

- The last digit of 1^{1000} is 1; 5^{1000} is 5; 6^{1000} is 6, 11^{1000} is 1. The results are easy to determine.
- A systematic listing is suggested for the other problems in the lesson.

2	3	Bonus
$2^1 \rightarrow 2$	$3^1 \rightarrow 3$	$6^1 \rightarrow 6$
$2^2 \rightarrow 4$	$3^2 \rightarrow 9$	$6^2 \rightarrow 36$
$2^3 \rightarrow 8$	$3^3 \rightarrow 7$	$6^3 \rightarrow 16$
$2^4 \rightarrow 6$	$3^4 \rightarrow 1$	$6^4 \rightarrow 96$
$2^5 \rightarrow 2$	$3^5 \rightarrow 3$	$6^5 \rightarrow 76$
$2^6 \rightarrow 4$	$3^6 \rightarrow 9$	$6^6 \rightarrow 56$
$2^7 \rightarrow 8$	.	$6^7 \rightarrow 36$
$2^8 \rightarrow 6$	.	$6^8 \rightarrow 16$
$2^9 \rightarrow 2$	.	$6^9 \rightarrow 96$
.	Every 4th number ends in 1. Thus, $3^{1000} \rightarrow 1$.	$6^{10} \rightarrow 76$
.		Every 5th number ends in 76. Thus, $6^{1000} \rightarrow 76$.
.		
Every 4th number ends in 6. Thus, $2^{1000} \rightarrow 6$.		

Answers:

1. 1	6. 6	11. 1	16. 1
2. 6	7. 1	12. 6	17. 1
3. 1	8. 6	13. 1	18. 0
4. 6	9. 1	14. 6	
5. 5	10. 0	15. 5	

Bonus:

76

BIRTHDAY PARTY

1. Chris is having a birthday party. The following clues should help you determine how many children were there.

   . Chris' mother brings a dish with 15 pieces of candy. She passes it around the table to all the children.

   . Each child takes a piece of candy, in turn, until the plate is empty.

   . Chris gets the first piece.

   . Chris also gets the next to the last piece.

   . Chris might also get other pieces as well.

   How many children could be at the party?

   (There are 2 possible answers to this puzzle. One solution could be that Chris is all alone.)

---

2. Suppose Chris' mother had brought a dish with 26 pieces of candy instead of 15. Chris still gets the first and next to last pieces. Now how many children could be present?

   (There are 8 possible answers.)

EXTENSION

The first puzzle had two solutions by starting with 15 pieces of candy. It's possible to start with other amounts of candy and still have only two solutions. See how many of these "starting amounts" you can find.

Birthday Party

Problem-solving skills pupils might use:

- Use a drawing.
- Use a systematic list.
- Guess and check.

Comments and suggestions:

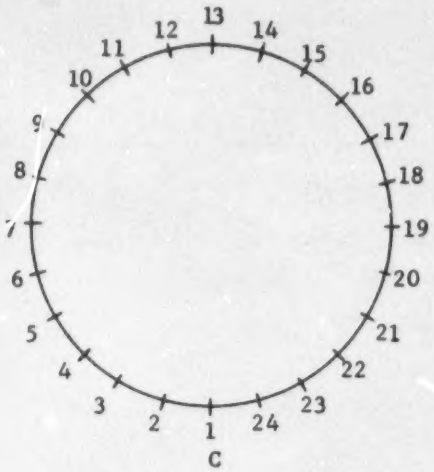
There are several different strategies pupils might use:

- One might be to actually act out the problem with a group of classmates using a guess and check procedure.
- Another could be to use a systematic "list." This one shows the solution to the first problem.

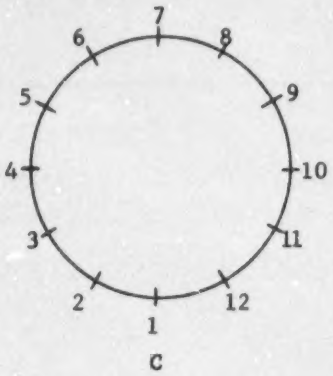
C x x x x x x x x x x x x | C x

Notice that - Chris gets the 1st and 2nd to last piece.
There are 15 pieces altogether.
13 children are in the group.

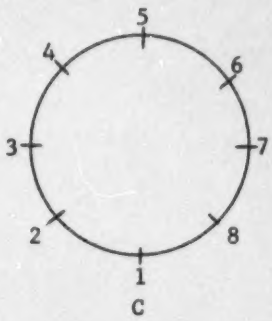
- Circular drawings like those below are helpful in analyzing problem 2.



24 children; Chris gets the 1st and 25th pieces.



12 children; Chris gets the 1st, 13th and 25th pieces.



8 children; Chris gets the 1st, 9th, 17th, and 25th pieces.

In like manner, you can make drawings to show that 6, 4, 3, 2, and 1 also are possible solutions. In fact, all the factors of 24 are solutions.

In general, the solutions are the factors of (n - 2), where n is the number of pieces of candy and $n \geq 2$.

Answers:

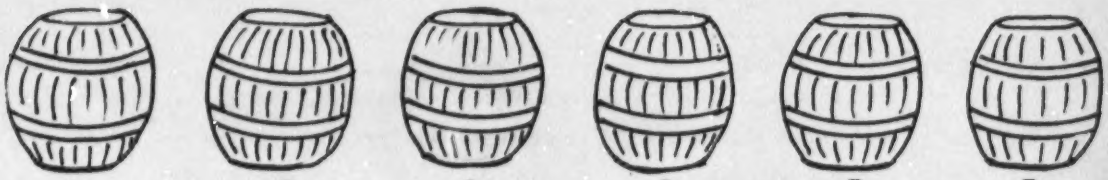
1. 13, 1 2. 24, 12, 8, 6, 4, 3, 2, 1

Extension: Two solutions are possible only when (n - 2) is prime.

84-

WHERE IS THE MONKEY?

1. There is a monkey hiding in one of the following barrels.



The monkey is in the 500th barrel.

Impossible, you say? Not so - the 6 barrels are counted as follows:

A	B	C	D	E	F
1	2	3	4	5	6
11	10	9	8	7	
	12	13	14	15	16
21	20	19	18	17	
	22	23	etc.		

In which barrel is the monkey hiding?

2. Suppose the barrels are numbered in the same way - but this time there are 4 barrels rather than 6.

   Now, where is the monkey hiding if he's still hiding in the 500th barrel?

3. Investigate other situations in which the monkey is hiding in the 500th barrel. For example, suppose there were 2 columns of barrels? or 3 columns of barrels? or 5 columns of barrels? etc. Describe any patterns you discover.

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Where Is the Monkey?

Problem-solving skills pupils might use:

- Make predictions based upon patterns.
- Use a systematic list.
- Eliminate possibilities.

Comments and suggestions:

- Pupils could solve the problems by extending the lists. Instead, encourage them to look for patterns within the lists.
- The pattern in the first problem is not difficult to discover. The monkey is in barrel B since all multiples of 10 are found in the B column.
- The second problem is not as easy. We see from the list at the right that the monkey could be in B or D. Notice also that the numbers in D begin with 4 and jump by 6 each time. Thus, if 500 is in column D, we must be able to express 500 as $4 + 6n$.
- Another pattern which appears in D is 10, 40, 70, 100 . . . If we continue to add 30, we see that 500 is not in D. Thus, the monkey must be in B.
- One final observation. If we divide each number in the list by 6, the remainders are as follows:

  Column A - Remainder 1
  Column B - Remainder 0 and 2
  Column C - Remainder 3 and 5
  Column D - Remainder 4

A	B	C	D
1	2	3	4
7	6	5	
	8	9	10
13	12	11	
	14	15	16
19	18	17	
	20	21	22
25	24	23	
	26	27	28
31	30	29	
	32	33	34
37	36	35	
	38	39	40
43	42	41	
	44	45	46
49	48	47	
	50	51	52
55	54	53	
	56	57	58
61	60	59	
	62	63	64
67	66	65	
	68	69	70

Answers:

1. Barrel B
2. Barrel B
3. Answers will vary.

A ROW OF PENNIES

Suppose that there are 200 pennies, all in a row. All of them are "heads" up.

Now, suppose that 200 people line up and -

- . the first person turns over all the pennies.
- . starting with the 2nd penny, the 2nd person turns over every other one.
- . starting with the 3rd penny, the 3rd person turns over every 3rd penny.
- . starting with the 4th penny, the 4th person turns over every 4th penny.

etc.

etc.

etc.

1. The first person will turn over the 200th penny. So will the second person. Altogether, 12 persons will turn over the 200th penny. Which persons are they?

2. Suppose you're the 200th person. Will you find the 200th penny "heads" or "tails?"

3. After everyone has gone through the line, which coins will show"tails?"

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A Row of Pennies

Problem-solving skills pupils might use:

- **Solve a related problem that is easier.**
- **Make a systematic list.**
- **Look for patterns.**

Comments and suggestions:

- **It is best to give this challenge during the time the class is studying number theory concepts.**
- **These problems can be solved by first looking at a simpler case - one in which both the number of coins and the number of people is 10.** A chart might be helpful in keeping track of the progression of the problem.

Pennies

People	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
1st	T	T	T	T	T	T	T	T	T	T
2nd		H		H		H		H		H
3rd			H			T			H	
4th				T				T		
5th					H					T
6th						H				
7th							H			
8th								H		
9th									T	
10th	T	H	H	T	H	H	H	H	T	H

Answers:

1. 1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 200.

   Note that for 10 people, the numbers are 1, 2, 5, and 10 (all the factors of 10). For 200, we find all the factors of 200.

2. You will find the 200th penny "tails." You will make it "heads." Note that in the chart, the 10th person will find the 1st, 4th, and 9th coins as "tails." These numbers are perfect squares. If we extend this pattern we see that 200 is not a perfect square--thus, it is "heads."

3. 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196 (all numbers from 1 to 200 that are perfect squares).

MR. WIZARD'S MARBLE TRICK

Mr. Wizard has a large collection of marbles. He counts out 45 of them and puts a different number in each bag. Billie, a fan of Mr. Wizard's, does not know how many are in each bag. But she does know that the total is 45 in all four.

Mr. Wizard then
- puts 2 more marbles in the first bag
- takes 2 marbles out of the second bag
- puts some marbles in the third bag so that it now contains twice as many
- takes enough marbles out of the fourth bag so that it now contains only half as many.

He then asks Billie to count the marbles in each bag.

"They all have the same number!" exclaims Billie.

1. How many marbles did Billie count in each bag?
2. How many marbles were in each bag at the start?

Suppose Mr. Wizard starts with 63 marbles rather than with 45. He performs the same "magic" as before.

3. Now how many marbles does Billie count in each bag?
4. How many were in each bag at the start?

EXTENSION

Are there other totals (besides 45 and 63) that work? Organize your results and look for patterns.

Mr. Wizard's Marble Trick

Problem-solving skills pupils might use:

- Guess and check.
- Use a systematic list.
- Work backwards.

Comments and suggestions:

- Pupils may have some difficulty understanding the problem until they try some numbers.
- To get them started, ask them to show why 10, 15, 8, and 12 are not the correct numbers of marbles to start with. A chart is one way of keeping track of their trials.

	1st Bag	2nd Bag	3rd Bag	4th Bag
Start	10	15	8	12
After Wizard	12	13	16	6

- After several trials, some pupils may discover a "working backwards" technique.

Answers:

1. 10
2. 8, 12, 5, and 20
3. 14
4. 12, 16, 7, and 28

Extension:

One way to summarize the solutions is to organize the results in a chart. The solutions for "45" and "63" are listed in the chart, as well as several others. In fact, there are an infinite number of totals that can be used.

Total	"Magic" Number	1st Bag	2nd Bag	3rd Bag	4th Bag
18	4	2	6	2	8
27	6	4	8	3	12
36	8	6	10	4	16
→ 45	10	8	12	5	20
54	12	10	14	6	24
→ 63	14	12	16	7	28

CHIPS AHOY

1. Alice, Bonita, and Carmen have just finished playing three rounds of a game. In each round there was only one loser:

   Alice lost the first round;
   Bonita lost the second;
   Carmen lost the third.

   After each round the loser was required to double the chips of each of the others by giving away some of her own chips.

   After three rounds each of the girls had 8 chips.

   How many chips did each girl have at the start?

2. Suppose each girl had 24 chips at the finish, rather than 8. Now how many chips did each girl have at the start?

EXTENSION

Show that it is impossible for each girl to have 20 chips at the finish.

Give another impossible situation.

Give some more possible situations.

Chips Ahoy

Problem-solving skills pupils might use:

- Work backwards.
- Guess and check.
- Use a systematic list.

Comments and suggestions:

- Pupils may have difficulty understanding the problem until they follow through a "trial run." To get them started, they could be asked to show why a certain number of chips will not work.
- A chart is one way of keeping track of the progression of their trials.

	A	B	C
At Start	30	16	4
End of 1st Game	10	32	8
End of 2nd Game	20	14	16
End of 3rd Game	40	28	-18 (?)

- It is obvious that these initial guesses do not work since the final row is not composed of 8s. After further unsuccessful trials, pupils may discover that the sum of each row must be 24.
- Also some pupils may discover a "working backwards" technique—start with 8's in the bottom row of the table and then work up to the beginning.

Answers:

1. Alice 13; Bonita 7; Carmen 4
2. Alice 39; Bonita 21; Carmen 12

Extension:

- By working backwards we find that for each girl to end up with 20 chips, the starting amounts would be: Alice, 32.5; Bonita, 17.5; Carmen, 10.
- There are an unlimited number of impossible situations. For example, it's never possible to end with each girl having an odd number of chips.
- There are infinitely many possible situations; i.e., when the numbers at the end are multiples of eight.

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